Embedding the Circle into the Plane in Unbounded KK-theory

Master's Thesis in Mathematical Physics.
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Abstract

We construct an unbounded representative for the shriek class associated to the embedding $\iota : S^1 \hookrightarrow \mathbb{R}^2$ as defined in [CS84] and equip this $KK_1(C(S^1), C_0(\mathbb{R}^2))$ cycle with a connection. Using this connection, we compute the product $\iota_! \otimes [\mathbb{R}^2]$ and prove that this provides an unbounded $KK_1(C(S^1), \mathbb{C})$ cycle. In this product we identify an index class in $KK_0(\mathbb{C}, \mathbb{C})$ which represents the multiplicative unit. We then show that the product $\iota_! \otimes [\mathbb{R}^2]$ represents the product of the fundamental class of S^1 with the index class using Kucerovsky's criterion [Kuc96].

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1 Introduction

In the 1930's Israel Gelfand showed that any commutative C^*-algebra is isometrically isomorphic to $C(X)$ for a locally compact Hausdorff topological space X. Conversely every locally compact Hausdorff topological space X induces a C^*-algebra $C(X)$, and these processes are inverse to each other, up to isomorphism or homeomorphism. This means that we can essentially do topology (of locally compact Hausdorff spaces) in terms of the continuous function algebra, instead of the usual point-set appraoch.

It turns out that many topological constructions, when formulated in terms of the corresponding commutative algebra, can be generalized to non-commutative algebras. One of these constructions is K-theory. In topological terms it studies the equivalence classes of vector bundles over the space X, which together with the Whitney sum of vector bundles form a semi-group.

The concept of vector bundles can be translated into algebraic language using the Serre-Swan theorem [Bla98, Ch. 1], by considering projections in the infinite matrix algebra $M_\infty(C(X))$. Since this construction using projections also makes sense in a non-commutative context we find a "topological invariant" associated to a non-commutative algebra.

Translating topological constructions into constructions on non-commutative algebras is the core idea of non-commutative geometry. However, so far we have been discussing topology rather than geometry. The crucial step to geometry is the inclusion of a Dirac operator into the data, as demonstrated by Alain Connes [Con94].

In this view of geometry we encode a manifold in terms of a commutative algebra of smooth functions, a Hilbert space of spinors and a Dirac operator as we further explore in Section 4.4. This paves the way for non-commutative geometry by replacing the smooth functions with a certain class of non-commutative algebras and combining them with Dirac-type operators.

These so called *spectral triples* are examples of KK classes, which in turn are elements of a generalization of K-theory called Kasparov's KK-theory. This opens up the application of several constructions from KK-theory to non-commutative geometry. Prime example of such a construction is the shriek class $f_!$ associated to a smooth map $f : M \to N$ as described in [CS84]. While [CS84] nominally deals with bounded KK-theory, the constructions are very much unbounded in character. This makes the shriek class especially interesting from a noncommutative geometry standpoint.

In [KS16] J. Kaad and W. van Suijlekom investigate this shriek class for submersions $\pi : M \to B$ in terms of unbounded KK-theory. The use of unbounded KK-theory is of interest here, since it allows us to investigate the geometric aspect of the immersion whereas bounded KK-theory is topological in nature. They construct an unbounded representative for the shriek class using a "vertical Dirac operator" D_π and prove that the spectral triple of the source manifold M factorizes into this vertical Dirac operator and the spectral triple of the base manifold B, up to a term coming from the curvature of the submersion.

Since the construction of the shriek class simplifies significantly in the cases of submersions and immersions relative to arbitrary maps, it is natural to now try to construct an unbounded representative for the shriek class $\iota_!$ of an immersion $\iota : M \to N$ and to prove analogous statements for $\iota_!$ as we did for submersions. In this thesis we will construct an unbounded representative for the shriek class of the immersion $\iota : S^1 \hookrightarrow \mathbb{R}^2$ which functions as a toy model for the general case.

Different from the case of submersions, where, essentially, new dimensions are added to the KK-cycles we need to remove dimensions. We find that the way to move down from the "large object" $\iota_! \otimes [\mathbb{R}^2]$ to the "small" object $[S^1]$ is an index class in $KK_0(\mathbb{C}, \mathbb{C})$. This is a new phenomenon compared to the submersion case, and plays a crucial rôle in our work.

The setup of this thesis is as follows. We spend the first chapters introducing the required concepts from KK-theory, starting with some preliminaries and moving on to the essentials of bounded and unbounded KK-theory. Then in chapter 5 we introduce our unbounded representative for the shriek class and prove the required analytical properties. As in [KS16] this construction is based on the bounded variant constructed in [CS84].

In chapter 6 we investigate a specific representative of the multiplicative unit in $KK_0(\mathbb{C}, \mathbb{C})$. This Index class is necessary to prove our analogue of the factorization result for submersions. It will turn out that we find that the Dirac operator on the product of $\iota_!$ and the triple of $\mathbb{R}^2$ is the Dirac operator of S^1 plus a term corresponding to this unit element.

Finally, in chapter 7 we prove that the shriek class $\iota_!$ indeed provides a KK-theoretic factorization of the spectral triple S^1 into $\iota_!$ and the spectral triple of $\mathbb{R}^2$ using unbounded KK-theoretic techniques from [Kuc96]. We actually already get this by proving that our unbounded representative indeed represents the shriek class, but proving this purely in an unbounded setting provides more geometric insight.

2 C^*-algebraic Preliminaries

In this section we will introduce several C^*-algebra related notions we will need to introduce KK-theory and prove our later results. We assume the reader has basic familiarity with C^*-algebras and Hilbert spaces and their (unbounded) operators, such as the background from Masters courses in Functional Analysis and Operator Algebras. We also assume basic familiarity with differential geometry, although this is only really used in Section 4.4.

Before we start let us introduce some notation.

Notation 2.1. *Let H be a Hilbert space, write $\mathcal{L}(H)$ for the bounded linear operators $H \to H$ and write $\mathcal{K}(H)$ for the compact operators $H \to H$.*

When writing tensor products of e.g. Hilbert spaces we write $H \otimes_{alg} K$ for the algebraic tensor product of H and K. Whenever we write $H \otimes K$ we mean the completion of the algebraic tensor product in the norm associated to the inner product $\langle \psi \otimes \phi, \xi \otimes \zeta \rangle = \langle \psi, \xi \rangle \langle \phi, \zeta \rangle$.

Whenever we take tensor products of C^*-algebra one of the components is finite dimensional. Therefore $A \otimes_{alg} B$ is already again a C^*-algebra and we simply write $A \otimes B$.

Finally, we write $\mathbb{C}\mathrm{l}_n$ for the Clifford algebra generated by $\mathbb{C}^n$ with quadratic form corresponding to the standard euclidean inner product, see e.g. [Sui15, Ch. 4.1] for the appropriate definitions.

2.1 Fredholm Operators

While Fredholm Operators will not make any explicit appearances outside of section 3.3, where we prove $KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, their properties provide some intuition behind the definitions in KK-theory. The most notable of these properties is the existence of an "Index", which (in various generalizations) pops up throughout K and KK theory.

Since we will use Fredholm operators only briefly, our introduction of them will be very concise. More comprehensive introductions can be found in e.g. [Mur14, Ch. 1.4] or probably your favourite textbook on functional analysis.

Definition 2.2. *Let H_1, H_2 be Hilbert spaces. A bounded, linear operator $F : H_1 \to H_2$ is called* Fredholm *if* $\dim(\ker(F))$ *and* $\dim(\mathrm{coker}(F))$ *are finite. Define the* Index *of F to be* $\mathrm{Index}(F) = \dim(\ker(F)) - \dim(\mathrm{coker}(F))$.

Theorem 2.3 (Atkinson)**.** *Let H be a Hilbert space and $F \in \mathcal{L}(H)$. Then F is Fredholm if and only if there exists $S \in \mathcal{L}(H)$ such that $1 - SF \in \mathcal{K}(H)$ and*

$1 - FS \in \mathcal{K}(H)$.

Proof. See [Mur14, Thm. 1.4.16] $\square$

The operator S appearing in Theorem 2.3 is called a *parametrix* for F. In the next section we will generalize the definition of Fredholm operators from Hilbert spaces to Hilbert C^*-modules using the existence of a parametrix.

We will now list some nice properties of Fredholm operators.

Proposition 2.4. *Let $F : H_1 \to H_2$ and $G : H_2 \to H_3$ be Fredholm and let $X \subset \mathcal{L}(H)$ be the set of all Fredholm operators. Then*

- *F^* is Fredholm and* $\operatorname{Index}(F^*) = -\operatorname{Index}(F)$.
- *GF is Fredholm and* $\operatorname{Index}(GF) = \operatorname{Index}(G) + \operatorname{Index}(F)$.
- *The map* $\operatorname{Index} : X \to \mathbb{Z}$ *is continuous relative to the norm topology.*
- *The* Index *is stable under compact perturbations.*
- *The* Index *of an invertible operator is 0, conversely a Fredholm operator with* Index *0 is a compact perturbation of an invertible operator.*

Proof. The proofs for all these statements can be found in [Mur14, Ch. 1.4]. $\square$

2.2 Hilbert C^*-modules

One of the main ingredients of a representative of a KK-class is a graded Hilbert bimodule, which we will introduce in this section mainly based on [Lan98, Ch. 3] and [Lan95] although we will generally use different notation. We will start by introducing Hilbert modules, then move on to Hilbert bimodules and finally we discuss gradings on both Hilbert bimodules and C^*-algebras.

A Hilbert C^*-module is essentially a generalization of a Hilbert space, where we replace the complex numbers by an arbitrary C^*-algebra. Many notions familiar from Hilbert spaces have analogues for Hilbert modules, although many theorems require various extra assumptions. Most of these assumptions are to deal with the fact that closed sub-modules may not be orthogonally complemented, as shown in Example 2.9.

Definition 2.5. *Let B be a C^*-algebra. A* pre-Hilbert B-module *consists of*

- *a complex linear space $\mathcal{E}$,*

- *a right-representation π of B on $\mathcal{E}$ as linear operators, i.e. for $\psi \in \mathcal{E}$, $b_1, b_2 \in B$*

$$\pi(b_1 b_2)\psi = \pi(b_2)\pi(b_1)\psi,$$

we will usually write $\pi(b)\psi$ as $\psi \cdot b$.

- *a map $\langle \cdot, \cdot \rangle_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \to B$ which is $\mathbb{C}$-linear in the second argument and conjugate $\mathbb{C}$-linear in the first, and satisfies*

$$\begin{aligned} \langle \psi, \phi \rangle_{\mathcal{E}}^* &= \langle \phi, \psi \rangle, \\ \langle \psi, \phi \cdot b \rangle_{\mathcal{E}} &= \langle \psi, \phi \rangle_{\mathcal{E}} b, \\ \langle \psi, \psi \rangle_{\mathcal{E}} &\geq 0, \\ \langle \psi, \psi \rangle_{\mathcal{E}} &= 0 \iff \psi = 0, \end{aligned}$$

with $\psi, \phi \in \mathcal{E}$ and $b \in B$. We call this sesquilinear form the inner product on $\mathcal{E}$.

If additionally $\mathcal{E}$ is complete in the norm $||\psi||_{\mathcal{E}}^2 := ||\langle \psi, \psi \rangle||_B$ we say that $\mathcal{E}$ is a Hilbert B-module*.*

It is common to build Hilbert modules out of pre-Hilbert modules, therefore the following Lemma is useful.

Lemma 2.6. *Any pre-Hilbert module can be completed into a Hilbert module.*

Proof. This is [Lan98, Cor. 3.2.4]. If $\mathcal{E}$ is a pre-Hilbert module, the completed Hilbert module is just the completion of $\mathcal{E}$ in the inner product norm with the representation and inner product extended by continuity. □

One of the first major differences between Hilbert modules and Hilbert spaces appears in their bounded operators. On a Hilbert module not all bounded maps have an adjoint, this leads us to the following definition.

Definition 2.7. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are Hilbert B-modules. A map $a : \mathcal{E} \to \mathcal{F}$ is called* adjointable *if there exists a map $a^* : \mathcal{F} \to \mathcal{E}$ such that*

$$\langle a\psi, \phi \rangle_{\mathcal{F}} = \langle \psi, a^* \phi \rangle_{\mathcal{E}}.$$

The set of all adjointable maps is denoted $\mathcal{L}(\mathcal{E}, \mathcal{F})$, if $\mathcal{F} = \mathcal{E}$ we write $\mathcal{L}(\mathcal{E})$ instead.

Adjointable maps are automatically bounded and B-linear and behave similar to the bounded linear operators on a Hilbert space. Indeed $\mathcal{L}(\mathcal{E})$ is a C^*-algebra.

Theorem 2.8. *Let $\mathcal{E}$ be a Hilbert B-module. Any element $a \in \mathcal{L}(\mathcal{E})$ is linear, B-linear and bounded. The adjoint is unique and defines an involution on $\mathcal{L}(\mathcal{E})$. When equipped with the operator norm $\mathcal{L}(\mathcal{E})$ is a C^*-algebra.*

Proof. This is [Lan98, Thm. 3.2.5]. Note that Landsman writes $C^*(\mathcal{E}, B)$ for $\mathcal{L}(\mathcal{E})$. □

We will now give an example of a Hilbert module with a bounded map that is not adjointable.

Example 2.9. *Let X be a compact Hausdorff space and $Y \subset X$ a non-empty closed subset with dense complement.*

The C^*-algebra $C(X)$ is a Hilbert $C(X)$-module with inner product $\langle f, g\rangle_{C(X)} = f^*g$ (in fact, any C^*-algebra A is a Hilbert A-module over itself with this inner product). Define $\mathcal{E} = \{f \in C(X) | f(Y) = 0\}$, this is also a Hilbert $C(X)$-module with inner product $\langle f, g\rangle_{\mathcal{E}} = f^*g$. For both Hilbert modules the right representation is simply multiplication on the right.

Consider $\mathcal{E}$ as a submodule of $C(X)$, we will show that $\mathcal{E}^{\perp\perp} \neq \mathcal{E}$ and $\mathcal{E} \oplus \mathcal{E}^{\perp} \neq C(X)$ even though $\mathcal{E}$ is closed.

Let $g \in \mathcal{E}^{\perp}$, then we must have $g(x) = 0$ for $x \in X \setminus Y$ so by continuity $g = 0$. Hence $\mathcal{E}^{\perp} = \{0\}$, so $\mathcal{E}^{\perp\perp} = C(X)$ and $\mathcal{E} \oplus \mathcal{E}^{\perp} = \mathcal{E}$.

This also breaks adjointability of maps. Let $\iota : \mathcal{E} \to C(X)$ be the inclusion map, which is clearly bounded and $C(X)$-linear. Suppose ι^* exists, and consider $\iota^*(1)$. Then for $x \in X$

$$\begin{aligned} f^*(x)\iota^*(1)(x) &= \langle f, \iota^*1\rangle_{\mathcal{E}}(x), \\ &= \langle \iota f, 1\rangle_{C(X)}(x), \\ &= f^*(x). \end{aligned}$$

This implies that $\iota^*(1)$ is constant 1 on $X \setminus Y$, so by continuity $\iota^*(1)$ would have to be 1 everywhere, but this is not an element of $\mathcal{E}$.

Besides an analogue of the bounded operators on a Hilbert space we would like an analogue of the compact operators on a Hilbert space. We do this using the property of Hilbert spaces that compact operators are limits of finite rank operators.

Definition 2.10. *Let $\mathcal{E}, \mathcal{F}$ be Hilbert B-modules and $\psi \in \mathcal{E}$, $\phi \in \mathcal{F}$. Define $|\phi\rangle\langle\psi| : \mathcal{E} \to \mathcal{F}$ by*

$$|\phi\rangle\langle\psi|\zeta = \phi \cdot \langle\psi, \zeta\rangle_{\mathcal{E}}.$$

This is an element of $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Define the set of finite rank operators *to be the linear span of these $|\phi\rangle\langle\psi|$.*

Lemma 2.11. *Let $\mathcal{E}, \mathcal{F}, \mathcal{E}'$ be Hilbert B-modules and $\psi \in \mathcal{E}$, $\phi, \xi \in \mathcal{F}$ and $\zeta \in \mathcal{E}'$. Furthermore, let $a \in \mathcal{L}(\mathcal{F}, \mathcal{E}')$, $b \in B$, $c \in \mathcal{L}(\mathcal{F})$ then*

- $(|\phi\rangle\langle\psi|)^* = |\psi\rangle\langle\phi|$,
- $|\zeta\rangle\langle\xi| \circ |\phi\rangle\langle\psi| = |\zeta \cdot \langle\xi, \phi\rangle_{\mathcal{F}}\rangle\langle\psi|$,
- $a|\phi\rangle\langle\psi| = |a\phi\rangle\langle\psi|$.

Proof. These all follow immediately from Definition 2.10. □

Definition 2.12. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. We define the set of compact operators between $\mathcal{E}$ and $\mathcal{F}$ as the norm-closure of the linear span of the finite rank operators and denote it $\mathcal{K}(\mathcal{E}, \mathcal{F})$. If $\mathcal{F} = \mathcal{E}$ we write $\mathcal{K}(\mathcal{E}) = \mathcal{K}(\mathcal{E}, \mathcal{E})$.*

Recall that the compact operators on a Hilbert space form an ideal in the bounded operators, the properties listed in Lemma 2.11 imply that this is also the case for Hilbert modules.

Finally, we need one technical property of Hilbert modules.

Definition 2.13. *A Hilbert B-module $\mathcal{E}$ is* countably generated *if there is a countable subset $E \subset \mathcal{E}$ such that $E \cdot B$ generates a dense subset of $\mathcal{E}$.*

We are now ready to move on to Hilbert bimodules. Since we have found $\mathcal{L}(\mathcal{E})$ as a generalization of $B(H)$, i.e. a C^*-algebra of operators on $\mathcal{E}$, it is natural to consider representations of other C^*-algebras on $\mathcal{E}$. This leads us to the definition of Hilbert bimodules.

Definition 2.14. *Let A and B be C^*-algebras. A* Hilbert A-B-bimodule *consists of a Hilbert B-module $\mathcal{E}$ together with a C^*-homomorphism $\Phi : A \to \mathcal{L}(\mathcal{E})$. We will usually suppress the representation Φ in our notation and write $\Phi(a)\psi = a \cdot \psi$ for $a \in A$ and $\psi \in \mathcal{E}$. We use ${}_A\mathcal{E}_B$ to denote a Hilbert A-B-bimodule.*

In [Lan98] Hilbert C^*-bimodules are instead called C^*-correspondences, hinting at the idea that Hilbert bimodules are in some sense "maps" between C^*-algebras. In any case, maps between C^*-algebras induce C^*-bimodules.

Example 2.15. *Let $\Phi : A \to B$ be a C^*-homomorphism. Define a Hilbert B-module by setting $\mathcal{E} = B$ with $\langle b_1, b_2\rangle_{\mathcal{E}} = b_1^* b_2$ and right-action by multiplication on the right. By the C^*-identity the norm on $\mathcal{E}$ coincides with the norm on B so that $\mathcal{E}$ is indeed complete.*

We make $\mathcal{E}$ into a Hilbert A-B module by defining a left-action by $a \cdot b = \Phi(a)b$. Denote this bimodule by ${}_\Phi B$.

This idea that bimodules are maps can be made more concrete, and is very relevant when considering for example Morita equivalance of C^*-algebras. However, it is not immediately relevant to our further discussion so we will simply use it as a motivation for certain constructions. One such construction is the balanced tensor product of bimodules, which corresponds to composition of maps.

Theorem 2.16. *Let A, B and C be C^*-algebras, and ${}_A\mathcal{E}_B$, ${}_B\mathcal{F}_C$ Hilbert bimodules. Define I_B to be the ideal in $\mathcal{E}\otimes_{alg}\mathcal{F}$ generated by elements of the form $\psi \cdot b \otimes \phi - \psi \otimes b \cdot \phi$. Then $(\mathcal{E}\otimes_{alg}\mathcal{F})/I_B$ is a pre-Hilbert C-module with right action*

$$[\psi \otimes \phi] \cdot c = [\psi \otimes \phi \cdot c]$$

and inner product

$$\langle [\psi \otimes \phi], [\xi \otimes \zeta] \rangle_{\mathcal{E}\otimes_B\mathcal{F}} = \langle \phi, \langle \psi, \xi \rangle_{\mathcal{E}} \cdot \zeta \rangle_{\mathcal{F}}.$$

Define $\mathcal{E} \otimes_B \mathcal{F}$ to be the completion of $(\mathcal{E}\otimes_{alg}\mathcal{F})/I_B$. Then $\mathcal{E} \otimes_B \mathcal{F}$ is a Hilbert C-module.

Furthermore, $\mathcal{L}(\mathcal{E})$ embeds in $\mathcal{L}(\mathcal{E} \otimes_B \mathcal{F})$ by $F \mapsto F \otimes 1$, i.e.

$$F(\psi \otimes \phi) = F(\psi) \otimes \phi.$$

Combining the representation of A with this embedding $\mathcal{E} \otimes_B \mathcal{F}$ becomes a Hilbert A-C-bimodule.

Proof. That $(\mathcal{E}\otimes_{alg}\mathcal{F})/I_B$ defines a pre-Hilbert C-module is proven in [Lan98, Prop. 4.5], then by Lemma 2.6 $\mathcal{E} \otimes_B \mathcal{F}$ is a Hilbert C-module. Since $F \in \mathcal{L}(\mathcal{E})$ is required to be B-linear, $F \otimes 1$ is well defined and it is an easy check that $F^* \otimes 1 = (F \otimes 1)^*$ so that $\mathcal{F} \otimes 1$ is indeed adjointable. $\square$

Remark 2.17. *We still write $\psi \otimes \phi$ for elementary tensors in $\mathcal{E} \otimes_B \mathcal{F}$ even though they really are equivalence classes of such elements.*

This product has many nice properties, which we will not prove as we do not rely on them later. For example, the product is associative up to unitary equivalence and for C^*-homomorphisms $\Phi : A \to B$, $\Psi : B \to C$ such that $\Psi(B)C$ is dense in C we have that ${}_{\Psi\circ\Phi}C$ is unitarily equivalent to ${}_\Phi B \otimes_B {}_\Psi C$.

Finally, we introduce gradings. First on C^*-algebras, then on Hilbert bimodules.

Definition 2.18. *Let A be a C^*-algebra. A* grading *is given by a C^*-automorphism $\gamma : A \to A$ satisfying $\gamma^2 = 1$.*

This induces a direct sum decomposition $A = A^0 \oplus A^1$ where $A^0 = \{a \in A|\gamma(a) = a\}$ and $A^1 = \{a \in A|\gamma(a) = -a\}$. Indeed we have $a = a_0 + a_1$ with $a_0 = \frac{1}{2}(a+\gamma(a))$, $a_1 = \frac{1}{2}(a - \gamma(a))$.

Elements $a \in A^i$ are called homogeneous of degree i. *We denote the degree of a homogeneous element $a \in A^i$ by $\partial a = i$. Note that $A^i A^j \subset A^{i+j \mod 2}$. We sometimes call A^0 the* even *part and A^1 the* odd *part of the C^*-algebra.*

When considering graded C^*-algebras the commutator is defined differently from what one might expect. For homogeneous elements we set

$$[a, b] = ab - (-1)^{\partial a \cdot \partial b} ba.$$

The commutator of general elements follows by linearity.

A homomorphism of graded C^*-algebras must preserve the grading, i.e. $\phi : A \to B$ is a homomorphism between graded C^*-algebras A and B whenever ϕ is a C^*-homomorphism and $\phi(A^i) \subset B^i$ (alternatively $\gamma \circ \phi = \phi \circ \gamma$). We say that a C^*-algebra is trivially graded if $\gamma = 1$ or equivalently $A^0 = A$.

Remark 2.19. *Whenever we deal with gradings we will use the "automorphism picture" and "decomposition picture" interchangeably since some properties are simply more convenient in one of the two pictures.*

Definition 2.20. *Let A and B be (possibly trivially) graded C^*-algebras. A graded Hilbert A-B bimodule is a regular Hilbert A-B bimodule $\mathcal{E}$ together with a linear operator $\gamma : \mathcal{E} \to \mathcal{E}$ such that $\gamma^2 = 1$ and*

- $\langle \gamma(\psi), \gamma(\phi) \rangle_{\mathcal{E}} = \gamma(\langle \psi, \phi \rangle_{\mathcal{E}})$,
- $\gamma(\psi \cdot b) = \gamma(\psi) \cdot \gamma(b)$.

In the decomposition picture using the ± 1-eigenspaces of γ as in Definition 2.18 these requirements are

- $\langle \mathcal{E}^i, \mathcal{E}^j \rangle_{\mathcal{E}} \subset B^{i+j \mod 2}$,
- $\mathcal{E}^i \cdot B^j \subset \mathcal{E}^{i+j \mod 2}$.

If $\mathcal{E}$ is graded then $\mathcal{L}(\mathcal{E})$ becomes a graded C^*-algebra by setting $\gamma(F)\psi = \gamma(F\gamma(\psi))$ for $F \in \mathcal{L}(\mathcal{E})$. In this grading we have $\mathcal{L}(\mathcal{E})^i \mathcal{E}^j \subset \mathcal{E}^{i+j \mod 2}$.

We then require the C^*-homomorphism $\Phi : A \to \mathcal{L}(\mathcal{E})$ from the bimodule to be a homomorphism of graded C^*-algebras.

Theorem 2.21. *Let A, B and C be graded C^*-algebras and $\mathcal{E}$ and $\mathcal{F}$ be graded Hilbert A-B and B-C bimodules respectively with grading operators $\gamma_{\mathcal{E}}$ and $\gamma_{\mathcal{F}}$.*

Then $\mathcal{E} \otimes_B \mathcal{F}$ from Theorem 2.16 with grading $\gamma_{\mathcal{E}} \otimes \gamma_{\mathcal{F}}$ defines a graded Hilbert A-C bimodule.

Proof. In the decomposition picture we find that

$$(\mathcal{E} \otimes_B \mathcal{F})^0 = \mathcal{E}^0 \otimes_B \mathcal{F}^0 \oplus \mathcal{E}^1 \otimes_B \mathcal{F}^1,$$
$$(\mathcal{E} \otimes_B \mathcal{F})^1 = \mathcal{E}^0 \otimes_B \mathcal{F}^1 \oplus \mathcal{E}^1 \otimes_B \mathcal{F}^0.$$

It is then a straightforward case-by-case verification that $\mathcal{E} \otimes_B \mathcal{F}$ is a graded Hilbert bimodule with grading $\gamma_{\mathcal{E}} \otimes \gamma_{\mathcal{F}}$. □

We will later have use for the following construction.

Definition 2.22. *Let ${}_A\mathcal{E}_B$ be a graded Hilbert bimodule, then the* opposite module *${}_A\mathcal{E}^{op}_B$ is the same linear space $\mathcal{E}$ but with grading $\gamma_{\mathcal{E}^{op}} = -\gamma_{\mathcal{E}}$ and representations $a \cdot \psi^{op} = (\gamma(a) \cdot \psi)^{op}$, $\psi^{op} \cdot b = (\psi \cdot b)^{op}$.*

2.3 Unbounded Operators

Unbounded operators on Hilbert modules suffer from the same problem as bounded operators. Where on a Hilbert space any densely defined closable operator has a densely defined adjoint, this fails for Hilbert modules. Although much theory carries over, it is usually with additional assumptions to guarantee existence, and in some sense well-behavedness, of the adjoint.

Definition 2.23. *Let $\mathcal{E}$, $\mathcal{F}$ be Hilbert B-modules. We write $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ for a B-linear map from a submodule $\mathrm{dom}(T) \subset \mathcal{E}$ to $\mathcal{F}$, we call $\mathrm{dom}(T)$ the* domain *of T. If it is clear in which module the domain of T lives we abbreviate this to $T : \mathrm{dom}(T) \to \mathcal{F}$.*

Let $T : \mathrm{dom}(T) \subset \mathcal{E}_1 \to \mathcal{E}_2$, $S : \mathrm{dom}(T) \subset \mathcal{E}_1 \to \mathcal{E}_2$ and $R : \mathrm{dom}(R) \subset \mathcal{E}_2 \to \mathcal{E}_3$ we write

- *$T \subset S$ if $\mathrm{dom}(T) \subset \mathrm{dom}(S)$ and $T\psi = S\psi$ for all $\psi \in \mathrm{dom}(T)$.*
- *$T + S : \mathrm{dom}(T + S) \to \mathcal{E}_2$ where $\mathrm{dom}(T + S) = \mathrm{dom}(T) \cap \mathrm{dom}(S)$ and $(T + S)\psi = T\psi + S\psi$ for all $\psi \in \mathrm{dom}(T + S)$.*
- *$RT : \mathrm{dom}(RT) \to \mathcal{E}_3$ where $\mathrm{dom}(RT) = \{\psi \in \mathrm{dom}(T) | T\psi \in \mathrm{dom}(R)\}$ and $(RT)\psi = R(T(\psi))$ for all $\psi \in \mathrm{dom}(RT)$.*

We say that $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ is densely defined *if $\mathrm{dom}(T)$ is dense in $\mathcal{E}$. We say that T is* closed *if*

$$G(T) := \{(\psi, T\psi) | \psi \in \mathrm{dom}(T)\} \subset \mathcal{E}_1 \oplus \mathcal{E}_2$$

is a closed submodule. We call $G(T)$ the graph *of T.*

We define the adjoint of an operator $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ the same way we do for Hilbert spaces.

Definition 2.24. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules and $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$. Define*

$$\mathrm{dom}(T^*) := \{\phi \in \mathcal{F} \mid \exists \xi \in \mathcal{E} : \langle T\psi, \phi\rangle_{\mathcal{F}} = \langle \psi, \xi\rangle_{\mathcal{E}} \forall \psi \in \mathrm{dom}(T)\}$$

If $\phi \in \mathrm{dom}(T^*)$ the corresponding ξ is unique and $T^*\phi = \xi$ defines a B-linear map.

For $v : \mathcal{E} \oplus \mathcal{F} \to \mathcal{F} \oplus \mathcal{E}$, $v(\psi, \phi) = (\phi, -\psi)$ we have $G(T^*) = vG(T)^\perp$, as in Hilbert spaces. But if we were working in Hilbert spaces we would have that if T were closed, T^* was densely defined and $G(T) \oplus vG(T^*) = \mathcal{E} \oplus \mathcal{F}$. On Hilbert modules, this is not necessarily true. This is the crucial difference between the theory on Hilbert modules and Hilbert spaces, tracing back to the possibility that closed subspaces are not orthogonally complemented we saw in Example 2.9. This is exactly what regularity is intended to fix.

Definition 2.25. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. An operator $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ is* regular *if T^* is densely defined and $1 + T^*T$ has dense range.*

Theorem 2.26. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. If $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ is a regular operator, then $G(T) \oplus v^*G(T^*) = \mathcal{E} \oplus \mathcal{F}$. Here $v : \mathcal{F} \oplus \mathcal{E} \to \mathcal{E} \oplus \mathcal{F}$, $(\phi, \psi) \mapsto (\psi, -\phi)$.*

Proof. This is [Lan95, Thm. 9.3]. □

Corollary 2.27. *If $T : \mathrm{dom}(T) \to \mathcal{F}$ is regular, it is closed.*

Proof. Suppose $\mathcal{E}$ is a Hilbert B-module and $X \oplus X^\perp = \mathcal{E}$. Then $X^{\perp\perp} = X$, so X is closed. If T is regular, $G(T) \oplus G(T)^\perp = G(T) \oplus v^*G(T^*) = \mathcal{E} \oplus \mathcal{F}$, so $G(T)$ is closed. □

There is also a partial converse.

Theorem 2.28. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert B-modules. If $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{F}$ is closed, T^* is densely defined and $G(T) \oplus v^*G(T^*)^\perp = \mathcal{E} \oplus \mathcal{F}$, then T is regular.*

Proof. This is [Lan95, Prop. 9.5] □

A useful, and for us essential, property of regular operators is the existence of a bounded transform.

Proposition 2.29. *Let $\mathcal{E}$ be a Hilbert B-module and $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{E}$ regular. Then $(1 + T^*T)^{-1}$ and $T(1 + T^*T)^{-\frac{1}{2}}$ extend to bounded operators in $\mathcal{L}(\mathcal{E})$.*

Proof. See [Bla98, Par. 13.3]. □

Definition 2.30. *If* $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{E}$ *is a regular operator we define the operator* $\mathfrak{b}(T)$ *as the extension of* $T(1+T^*T)^{-\frac{1}{2}}$. *We call* $\mathfrak{b}(T)$ *the* bounded transform *of* T.

Finally we will need to know what it means for an operator $T : \mathrm{dom}(T)\mathcal{E} \to \mathcal{E}$ to be self-adjoint. Fortunately this is again the same definition as for Hilbert spaces.

Definition 2.31. *Let* $\mathcal{E}$ *be a Hilbert* B*-module. We say that* $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{E}$ *is* symmetric *if* $T \subset T^*$ *and* self-adjoint *if* $T = T^*$. *We say that a closable operator* T *is* essentially self-adjoint *if the closure of* T *is self-adjoint.*

The main result we will use to prove self-adjointness of operators carries over directly from the Hilbert space setting.

Theorem 2.32. *Let* $\mathcal{E}$ *be a Hilbert* B*-module and suppose* $T : \mathrm{dom}(T) \subset \mathcal{E} \to \mathcal{E}$. *Then* T *is regular and essentially self-adjoint if* T *is symmetric and* $T \pm i : \mathrm{dom}(T) \to \mathcal{E}$ *have dense range.*

Proof. Since $(1 + T^*T) = (T^* + i)(T - i) \supset (T + i)(T - i)$, it has dense range so T is regular. Essential self-adjointness now follows with the same proof as in the Hilbert space setting which can be found in e.g. [Lax02, Thm 33.2]. $\square$

3 Introduction to bounded KK-theory

3.1 Definition of $KK_i(A, B)$

We will now introduce, often without proof, the essential definitions and results from KK-theory that are required for our later analysis. There are many reasons and ways to do KK-theory, but we will only provide the framework that is useful for the applications we need. A more complete and detailed introduction can be found in e.g. [Bla98] or [Hig90]. As a standing assumption we assume all C^*-algebras are separable and all Hilbert C^*-modules are countably generated.

Definition 3.1. *Let A, B be (possibly graded) C^*-algebras. The set $kk_0(A, B)$ is defined to be the set of triples $(\mathcal{E}, F)$ where*

- *$\mathcal{E}$ is a graded A-B Hilbert C^*-module,*
- *$F : \mathcal{E} \to \mathcal{E}$ is an bounded adjointable operator of degree 1,*
- *$a(F^2 - 1)$, $a(F^* - F)$ and $aF - Fa$ are compact.*

We call $(\mathcal{E}, F) \in kk_0(A, B)$ an even Kasparov A-B cycle *or A-B cycle for short.*

Remark 3.2. *Later on we will also encounter the notion of an odd Kasparov A-B cycle. We will usually omit the qualifier even/odd from our language for brevity and trust that it is clear from the context whether the cycle is even or odd.*

Definition 3.3. *Two Kasparov A-B cycles $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$ are unitarily equivalent if there is a unitary map $U : \mathcal{E}_0 \to \mathcal{E}_1$ of graded Hilbert A-B bimodules such that $UF_0U^* = F_1$.*

A homotopy between even Kasparov A-B cycles $(\mathcal{E}_0, F_0)$ *and $(\mathcal{E}_1, F_1)$ is given by an even A-$B \otimes C([0, 1])$ cycle $(\mathcal{E}, F)$ such that $(\mathcal{E} \otimes_{(B \otimes C([0,1]),\rho_i)} B, F \otimes 1)$ is unitarily equivalent to $(\mathcal{E}_i, F_i)$ for $i = 0, 1$. Here $\rho_i(b \otimes f)b' = f(i)bb'$ for $b, b' \in B$ and $f \in C([0, 1])$ and corresponds roughly to "evaluation at i".*

Definition 3.4. *Define $KK_0(A, B)$ to be $kk_0(A, B)/\sim$ where $\sim$ denotes homotopy equivalence. Note that in our definition homotopy equivalence includes unitary equivalence.*

Proposition 3.5. *The set $KK_0(A, B)$ is an abelian group with direct sum as operation. An additive inverse for $(\mathcal{E}, F)$ is $(\mathcal{E}^{op}, -F)$.*

Proof. See [Bla98, Prop 17.3.3]. □

Definition 3.6. *A Kasparov A-B cyle $(\mathcal{E}, F)$ is called* degenerate *if $a(F^2 - 1) = a(F^* - F) = aF - Fa = 0$ for all $a \in A$.*

Lemma 3.7. *Any degenerate Kasparov cycle represents* 0 *in the group* $KK_0(A, B)$.

Proof. We include the proof of this Lemma since it contains an interesting idea, called an "Eilenberg swindle" [HR00, Prop. 8.2.8].

Let $(\mathcal{E}, F)$ be a degenerate A-B cycle. Define $\mathcal{E}_\infty = \oplus_{i=1}^{\infty}\mathcal{E}$ and $F_\infty((e_i)_{i\in\mathbb{N}}) = (Fe_i)_{i\in\mathbb{N}}$. We claim that $(\mathcal{E}_\infty, F_\infty)$ again defines a Kasparov A-B cycle. Certainly $\mathcal{E}_\infty$ is still a graded A-B Hilbert C^*-module and $\mathcal{F}_\infty$ is a bounded adjointable operator, $(\mathcal{F}_\infty)^* = \mathcal{F}_\infty^*$. Furthermore, $a(F_\infty^2 - 1) = \oplus_{i\in\mathbb{N}}a(F^2 - 1) = \oplus_{i\in\mathbb{N}}0 = 0$, and the same holds $aF - Fa$ and $a(F^* - F)$ so that $(\mathcal{E}_\infty, F_\infty)$ still defines a Kasparov A-B cycle.

Now clearly $(\mathcal{E}, F) \oplus (\mathcal{E}_\infty, F_\infty)$ is unitarily equivalent to $(\mathcal{E}_\infty, F_\infty)$, so they represent the same element in $KK(A, B)$. Since $KK(A, B)$ is a group with operator $\oplus$ by Proposition 3.5, this implies that $(\mathcal{E}, F)$ represents the additive unit. $\square$

Proposition 3.8. *The construction of* $KK_0(A, B)$ *defines a homotopy invariant bifunctor from pairs of* C^**-algebras to abelian groups. This functor is contravariant in the first variable and covariant in the second variable.*

Proof. See [Bla98, Par. 17.8, 17.9]. $\square$

A useful result by Kasparov allows us to construct $KK_0(A, B)$ using a simpler notion of homotopy than the one from Definition 3.3.

Definition 3.9. *An* operator homotopy *between two Kasparov cycles* $(\mathcal{E}, F_0)$ *and* $(\mathcal{E}, F_1)$ *is given by a norm-continuous map* $[0, 1] \ni t \mapsto F_t$ *such that* $(\mathcal{E}, F_t)$ *is a Kasparov cycle for all* t *and* $F_{t=0} = F_0$*,* $F_{t=1} = F_1$*.*

Proposition 3.10. *Two cycles* $(\mathcal{E}_0, F_0)$ *and* $(\mathcal{E}_1, F_1)$ *are homotopic in the sense of Definition 3.3 if and only if they are homotopic in the sense of Definition 3.9 up to addition of degenerate Kasparov cycles.*

Proof. See [Bla98, Par. 18.5], note that this proof uses the Kasparov product which we define later. $\square$

The notion of operator homotopy makes it easy to state the following Corollary, although it is also an immediate consequence of the definition of homotopy in Definition 3.3.

Corollary 3.11. *Let* $(\mathcal{E}, F)$ *be a Kasparav* A-B *cycle and* $K : \mathcal{E} \to \mathcal{E}$ *a compact operator. Then* $(\mathcal{E}, F)$ *and* $(\mathcal{E}, F + K)$ *represent the same class in* $KK_0(A, B)$.

Proof. An operator homotopy is given by $[0, 1] \ni t \mapsto F + tK$. $\square$

There is a nice connection between KK-theory and the "ordinary" K-groups.

Proposition 3.12. *We have $KK_0(\mathbb{C}, B) \cong K_0(B)$ and $KK_0(A, \mathbb{C}) \cong K^0(A)$.*

Proof. For the first assertion, see [Bla98, Prop. 17.5.5]. For the second assertion we refer to [HR00, Ch. 8.4]. □

The motivation for the 0 in $KK_0(A, B)$ comes from the existence of higher order KK groups, similar to the existence of higher order K groups. We even have a notion similar to Bott periodicity.

Definition 3.13. *We define the higher order KK-groups by*

$$KK_n(A, B) = KK(A, B \otimes \mathbb{C}\mathrm{l}_n),$$
$$KK_{-n}(A, B) = KK(A \otimes \mathbb{C}\mathrm{l}_n, B)$$

for $n \in \mathbb{N}$.

Proposition 3.14. *We have $KK_l(A, B) \cong KK_{l+2}(A, B)$ for all $l \in \mathbb{Z}$.*

Proof. See [Ech17, Proposition 3.23]. □

Proposition 3.14 tells us that we lose no information restricting to $KK_0(A, B)$ and $KK_1(A, B)$ or $KK_{-1}(A, B)$. Elements of $KK_{\pm 1}(A, B)$ are, in principle, represented by *even* A-$B \otimes \mathbb{C}\mathrm{l}_1$ or $A \otimes \mathbb{C}\mathrm{l}_1$-$B$ cycles, however we may also represent them using *odd* cycles, although we defer the proof of this claim until we treat the unbounded version.

Definition 3.15. *An* odd *Kasparov A-B cycle is a pair $(\mathcal{E}, F)$ where $\mathcal{E}$ is a A-B Hilbert bimodule and $F : \mathcal{E} \to \mathcal{E}$ a bounded adjointable map such that $a(F^2 - 1)$, $aF - Fa$ and $a(F^* - F)$ are compact.*

Remark 3.16. *The only difference with Definition 3.1 is that $\mathcal{E}$ is ungraded and F therefore not required to be odd.*

3.2 The Kasparov Product

One of the main tools that Kasparov's KK-groups provide is the Kasparov product, which is a map $KK_0(A, B) \times KK_0(B, C) \to KK_0(A, C)$. A downside of the Kasparov product is that an explicit formula for the product of two cycles is hard to give in general, therefore we only give a criterion for an A-C cycle $(\mathcal{E}, F)$ to represent the product of an A-B cycle $(\mathcal{E}_1, F_1)$ and a B-C cycle $(\mathcal{E}_2, F_2)$. In order to introduce this we will follow [Con94, Ch. 4, Appendix 1] but refer to [Bla98] for proofs.

Definition 3.17. *Let $\mathcal{E}_1$ be an Hilbert B-module and $\mathcal{E}_2$ a Hilbert B-C bimodule and $\mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E}_2$. Define for $\xi \in \mathcal{E}_1$ the map $T_\xi : \mathcal{E}_2 \to \mathcal{E}$, $T_\xi(\eta) = \xi \otimes \eta$ for all*

$\eta \in \mathcal{E}_2$. *The adjoint* T_ξ^* *is given by* $T_\xi^*(\nu \otimes \eta) = \langle \xi, \nu \rangle_B \cdot \eta$ *on elementary tensors* $\nu \otimes \eta \in \mathcal{E}$.

Definition 3.18. *Let* $\mathcal{E}_1$ *be a Hilbert B-module, and* $(\mathcal{E}_2, F_2)$ *a Kasparov B-C module. Set* $\mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E}_2$ *then* $F : \mathcal{E} \to \mathcal{E}$ *is an* F_2-connection *on* $\mathcal{E}_1$ *if*

$$\left[\begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix}, \begin{pmatrix} F_2 & 0 \\ 0 & F \end{pmatrix} \right]$$

is compact on $\mathcal{E}_2 \oplus \mathcal{E}$ *for all* $\xi \in \mathcal{E}_1$.

Definition 3.19. *Suppose* $(\mathcal{E}_1, F_1)$ *and* $(\mathcal{E}_2, F_2)$ *are even Kasparov A-B, B-C cycles respectively. Let* $\mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E}_2$ *and let* $F : \mathcal{E} \to \mathcal{E}$. *Then we say that* $(\mathcal{E}, F)$ *represents* the Kasparov product *of* $(\mathcal{E}_1, F_1)$ *and* $(\mathcal{E}_2, F_2)$ *if*

- $(\mathcal{E}, F)$ *is an even Kasparov A-C cycle,*
- F *is an* F_2 *connection, (connection condition)*
- *For all* $a \in A$ *there is a compact* $K : \mathcal{E} \to \mathcal{E}$ *such that* $a[F_1 \otimes 1, F]a^* + K \geq 0$. *(positivity condition)*

We write $(\mathcal{E}, F) \in (\mathcal{E}_1, F_1)\#(\mathcal{E}_2, F_2)$.

In order to show that this definition for the Kasparov product makes sense, we include the following two propositions.

Proposition 3.20. *Let* $(\mathcal{E}_1, F_1)$ *and* $(\mathcal{E}_2, F_2)$ *be Kasparov A-B, B-C cycles respectively. There exists a Kasparov product* $(\mathcal{E}, F)$, *and any two products represent the same class in* $KK_0(A, C)$.

Proof. See [Bla98, Thm. 18.4.3]. This is a consequence of Kasparov's Technical Theorem. □

Proposition 3.21. *The Kasparov product defines a bilinear product* $KK_0(A, B) \times KK_0(B, C) \to KK_0(A, C)$. *In particular,* $KK_0(A, A)$ *is a ring.*

Proof. See [Bla98, Thm. 18.4.4]. To see that $KK_0(A, A)$ forms a ring only associativity of the product and existence of a unit are missing Associativity is proven in [Bla98, Thm. 18.6.1], a representative for the multiplicative unit is $(A, 0)$ with trivial grading [Ech17, Thm. 3.12]. □

Remark 3.22. *In general we have a Kasparov product* $KK_i(A, B) \times KK_j(B, C) \to KK_{i+j}(A, C)$ *whose construction follows completely from the* KK_0 *case.*

3.3 The Index Pairing: $KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$

One of the aspects of ordinary K (co)homology that appears nicely in KK theory is the index pairing between $K^0(A)$ and $K_0(A)$. This index pairing is a bilinear map $K_0(A) \times K^0(A) \to \mathbb{Z}$. Using the Kasparov product and Proposition 3.12 this takes the form $KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \to KK_0(\mathbb{C}, \mathbb{C})$. In this section we will show that $KK_0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}$, not just as group but as ring.

Before we start let us note two useful simplification results.

Proposition 3.23. *Any class in $KK_0(A, B)$ can be represented by a cycle $(\mathcal{E}, F)$ where F is exactly self-adjoint.*

Proof. See [Bla98, Prop. 17.4.2] for all details, but the central idea is that $(\mathcal{E}, \frac{1}{2}(F + F^*))$ is also a Kasparov A-B cycle. $\square$

Proposition 3.24. *Suppose A is unital. Any class in $KK_0(A, B)$ can be represented by cycle $(\mathcal{E}, F)$ where the representation of A on $\mathcal{E}$ is unital.*

Proof. See [Bla98, Par. 17.5] for all details, but the central idea is to replace $(\mathcal{E}, F)$ by $(1 \cdot \mathcal{E}, 1F1)$. $\square$

Let us first investigate the definition of Kasparov cycles for $A = B = \mathbb{C}$. Going by the bullets of Definition 3.1 we get, also assuming the simplifications from Propositions 3.23 and 3.24,

- $\mathcal{E}$ is a graded Hilbert space,
- F is a bounded, self-adjoint, linear operator of degree 1,
- $F^2 - 1$ is compact.

Since we are, apparently, dealing with Hilbert spaces rather than modules we will usually denote Kasparov $\mathbb{C}$-$\mathbb{C}$ cycles by (H, F). Note that the third point implies that F is a Fredholm operator. Furthermore, we may split our Hilbert space $H = H^0 \oplus H^1$ into the even and odd component. With respect to this decomposition we have

$$F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$$

where $F_- = F_+^*$ since F is self-adjoint. Furthermore

$$F^2 - 1 = \begin{pmatrix} F_+^* F_+ - 1 & 0 \\ 0 & F_+ F_+^* - 1 \end{pmatrix}$$

which means that also F_+ is Fredholm, with parametrix F_+^*.

Definition 3.25. *Let (H, F) be a Kasparov $\mathbb{C}$-$\mathbb{C}$ cycle. Then as above, $F = \left(\begin{smallmatrix} 0 & F_- \\ F_+ & 0 \end{smallmatrix}\right)$. Define* the index of (H, F) *as* $\mathrm{Index}((H, F)) = \mathrm{Index}(F_+)$, *i.e. the Fredholm index of F_+.*

Proposition 3.26. *The map* Index : $KK_0(\mathbb{C}, \mathbb{C}) \to \mathbb{Z}$ *is a well-defined group homomorphism.*

Proof. Let us start with well-definedness of this map. Suppose (H, F) and (H', F') represent the same class in $KK_0(\mathbb{C}, \mathbb{C})$, by Proposition 3.10 we may assume that (H, F) and (H', F') are operator homotopic up to addition of degenerate cycles. Therefore it is sufficient to show that

1. unitary equivalence,
2. operator homotopies,
3. addition of degenerate cycles,

preserve the index.

Since unitary equivalences between (H, F) and (H', F') preserve the grading on H and H' these unitaries actually intertwine F_+, F'_+ and F_-, F'_-. So we get a unitary U_+ such that $F_+ = U_+ F'_+ U_+^*$. But then $\ker(F_+) = U_+ \ker(F'_+) U_+^*$ and similar for $\ker(F_+^*)$ so that the Fredholm indices of F_+ and F'_+ coincide.

Now let $[0, 1] \ni t \mapsto F_t$ be an operator homotopy. Then we also get a norm-continuous map $[0, 1] \ni t \mapsto (F_t)_+$. The map $S \mapsto \mathrm{Index}(S)$ from Fredholm operators to $\mathbb{Z}$ is continuous (2.4), so the map $[0, 1] \ni t \mapsto \mathrm{Index}((F_t)_+)$ is continuous, hence constant.

To prove invariance under addition of degenerate cycles we first argue that degenerate cycles have index 0, and then that the index is additive. Suppose that (H, F) is a degenerate $KK_0(\mathbb{C}, \mathbb{C})$ cycle, i.e. $F^2 - 1 = 0$. Then $F_+^* F_+ = 1$ as well, so that F_+ is invertible. But then $\mathrm{Index}((H, F)) = \mathrm{Index}(F_+) = 0$ since invertible operators have index 0.

Finally we turn to additivity. Let (H, F) and (H', F') both be Kasparov $\mathbb{C}$-$\mathbb{C}$ cycles, then $(H, F) + (H', F') = (H \oplus H', F \oplus F')$. Decomposing $H \oplus H' = H^0 \oplus H'^0 \oplus H^1 \oplus H'^1$ yields

$$F \oplus F' = \begin{pmatrix} 0 & 0 & F_- & 0 \\ 0 & 0 & 0 & F'_- \\ F_+ & 0 & 0 & 0 \\ 0 & F'_+ & 0 & 0 \end{pmatrix}.$$

So $(F \oplus F')_+ = F_+ \oplus F'_+$ and $\mathrm{Index}(F_+ \oplus F'_+) = \mathrm{Index}(F_+) + \mathrm{Index}(F'_+)$. $\square$

Our next step is to show that the index map is actually an isomorphism. For this we need of course surjectivity and injectivity, of these injecitivity is the hard part so we start with that.

Proposition 3.27. *Let (H, F) be a Kasparov $\mathbb{C}$-$\mathbb{C}$ cycle with* $\mathrm{Index}((H, F)) = 0$*, then (H, F) represents 0.*

Proof. Our strategy will be to make a series of compact perturbations of (H, F) which lead to a degenerate cycle. Then Corollary 3.11 implies that (H, F) represents 0. Assume F is self-adjoint in accordance to Proposition 3.23.

By assumption $\mathrm{Index}(F_+) = 0$, then there is a compact operator K such that $F_+ + K$ is invertible ([Mur14, Rem. 1.4.3]). Define $F_1 = \begin{pmatrix} 0 & (F_+ + K)^* \\ F_+ + K & 0 \end{pmatrix}$ and $F_2 = F_1(F_1^* F_1)^{-\frac{1}{2}}$.

Let us consider $(F_1^* F_1)^{-\frac{1}{2}}$ for a moment. Expanding the F_1 yields

$$F_1^* F_1 = F^* F + F^* K + K^* F + K^* K.$$

The right hand side of this equation is a compact perturbation of 1, since $F^* F - 1 = F^2 - 1$ is compact. But then $(F_1^* F_1)^{-\frac{1}{2}}$ is also $1 + K'$ for some compact operator K'. This can be seen by considering $\pi : B(H) \to B(H)/K(H)$ which is a homomorphism, hence if $\pi(F_1^* F_1) = \pi(1)$ the same holds for their square roots.

Therefore F_2 is a compact perturbation of F_1, and hence also of F. The class represented by (H, F_2) is degenerate, since $F_2^2 = 1$ and F_2 is self-adjoint. But since (H, F_2) and (H, F) represent the same class in $KK(\mathbb{C}, \mathbb{C})$ we find that (H, F) represents 0. □

Proposition 3.28. *The map* $\mathrm{Index} : KK_0(\mathbb{C}, \mathbb{C}) \to \mathbb{Z}$ *is a group isomorphism.*

Proof. We have that Index is a homomorphism by Proposition 3.26 and we have injectivity by Proposition 3.27.

For surjectivity, consider $H = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ with grading $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and operator F_+ the right shift, i.e. $F_+\psi(0) = 0$, $F_+\psi(n + 1) = \psi(n)$. Then $F = \begin{pmatrix} 0 & F_+^* \\ F_+ & 0 \end{pmatrix}$ makes (H, F) into a Kasparov $\mathbb{C}$-$\mathbb{C}$ cycle with $\mathrm{Index}((H, F)) = 1$. Since the generator of $\mathbb{Z}$ lies in the range of Index, the Index is surjective. □

To extend from a group isomorphism to an isomorphism of rings is surprisingly easy.

Lemma 3.29. *The data $(\mathbb{C}, 0)$ with $\mathbb{C}$ trivially graded defines a multiplicative unit for $KK_0(\mathbb{C}, \mathbb{C})$.*

Proof. Note that, somewhat counter-intuitively, $(\mathbb{C}, 0)$ is not degenerate since $0^2 - 1 \neq 0$. It is very easy to show that all conditions in Definition 3.19 are satisfied for $\mathcal{E} = \mathcal{E}_1$ and $F = F_1$ in the case $(\mathcal{E}_1, F_1) \otimes_{\mathbb{C}} (\mathbb{C}, 0)$ and $\mathcal{E} = \mathcal{E}_2$, $F = F_2$ in the case $(\mathbb{C}, 0) \otimes_{\mathbb{C}} (\mathcal{E}_2, F_2)$.

We will also compute $\mathrm{Index}((\mathbb{C}, 0))$. Note that $0_+ = 0 : \mathbb{C} \to \{0\}$, so that $\mathrm{Index}(0_+) = \ker(0) - \mathrm{coker}(0) = 1$. $\square$

Corollary 3.30. *$KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ as rings.*

Proof. Let $\mathbb{1}$ be the multiplicative unit in $KK_0(\mathbb{C}, \mathbb{C})$, then $\mathrm{Index}(\mathbb{1}) = 1$ by Proposition 3.26 and the above Lemma. Hence any class in $x \in KK_0(\mathbb{C}, \mathbb{C})$ can be represented by a direct sum of $\mathrm{Index}(x)$ copies of $\mathbb{1}$, then distributivity of the Kasparov product shows that Index is also multiplicative. $\square$

4 Unbounded KK theory

Now that we have a basic idea what bounded KK-theory is, it is time to turn to unbounded KK-theory. The unbounded approach to KK-theory was introduced by Baaj en Julg [BJ83] and is the main approach we use in this thesis. It turns out to lend itself well to the computation of Kasparov products, as we will see in Sections 4.2 and 4.3.

4.1 Definitions

Let us first define the unbounded representatives of KK-classes.

Definition 4.1. *Let A and B be (possibly graded) C^*-algebras. We say that $({}_A\mathcal{E}_B, D : \mathrm{dom}(D) \subset \mathcal{E} \to \mathcal{E})$ is an* even unbounded Kasparov A-B cycle *if*

- *${}_A\mathcal{E}_B$ is a graded Hilbert bimodule and T is odd,*
- *D is self-adjoint and regular,*
- *$a(1 + D^2)^{-1} \in \mathcal{K}(\mathcal{E})$ for all $a \in A$,*
- *For all elements a of a dense subset $\mathcal{A} \subset A$ the graded commutator $[D, a]$ is defined on $\mathrm{dom}(D)$ and extends to a bounded operator on $\mathcal{E}$.*

We write $\Psi_0(A, B)$ for the set of all even unbounded Kasparov A-B cycles.

The "raison d'être" for this definition is the following theorem, due to Baaj and Julg [BJ83].

Theorem 4.2. *Let $({}_A\mathcal{E}_B, D) \in \Psi_0(A, B)$ be an unbounded Kasparov cycle. Then $({}_A\mathcal{E}_B, \mathfrak{b}(D))$ is a bounded Kasparov cycle, where $\mathfrak{b}(D)$ denotes the bounded transform from Definition 2.30. Moreover, the map $\Psi_0(A, B) \to KK(A, B)$, $(\mathcal{E}, D) \mapsto [(\mathcal{E}, \mathfrak{b}(D))]$ is surjective.*

Proof. This was first shown in [BJ83], the original paper by Baaj and Julg, but it does not contain a full proof. All details can be found in [Bla98, Thm. 17.11.3, 17.11.4]. □

Unbounded KK-theory is very interesting from the standpoint of noncommutative geometry, since unbounded Kasparov cycles are almost spectral triples. Indeed, suppose A is unital and trivially graded and $B = \mathbb{C}$. Then the requirements for an unbounded Kasparov cycle become

- ${}_A\mathcal{E}_{\mathbb{C}}$ is a graded Hilbert space with an even representation of A and an odd operator D,

- D is self-adjoint (regularity is automatic for Hilbert spaces),
- D has compact resolvent,
- for $a \in \mathcal{A}$ the commutator $[D, a]$ is bounded.

Hence $(\mathcal{A}, \mathcal{E}, D)$ is a spectral triple.

In Definition 3.15 we saw the definition of *odd* bounded Kasparov cycles. Similarly we have *odd* unbounded Kasparov cycles.

Definition 4.3. *Let A and B be trivially graded C^*-algebras. We say that $({}_A\mathcal{E}_B, D)$ is an odd unbounded Kasparov A-B cycle if ${}_A\mathcal{E}_B$ is a trivially graded Hilbert bimodule and the pair satisfies all conditions in Definition 4.1 except that D is odd.*

Denote the set of all odd unbounded Kasparov A-B cycles by $\Psi_1(A, B)$.

Since the bounded transform is surjective $\Psi_0(A, B) \to KK_0(A, B)$ it is also surjective $\Psi_0(A \otimes \mathbb{C}\mathrm{l}_1, B) \to KK_1(A, B)$. We now show, using work from [Dun16], that elements of $\Psi_0(A \otimes \mathbb{C}\mathrm{l}_1, B)$ can equivalently be described using odd unbounded Kasparov cycles.

Lemma 4.4. *Let $(\mathcal{E}, D)$ be an odd unbounded Kasparov A-B cycle. Then $(\mathcal{E} \otimes \mathbb{C}^2, D \otimes \gamma^1)$ is an even unbounded Kasparov $A \otimes \mathbb{C}\mathrm{l}_1$-$B$ cycle with grading $1 \otimes \gamma^3$ and $a \otimes (\alpha 1 + \beta e)$ acting via $\alpha a \otimes 1 + \beta a \otimes \gamma^2$, where e denotes the generator of $\mathbb{C}\mathrm{l}_1$ and the γ matrices are the Pauli matrices. We call this "doubling" an odd cycle.*

Proof. It is clear that $D \otimes \gamma^1$ is odd relative to $1 \otimes \gamma^3$, self-adjointness, regularity, compact resolvent and bounded commutators all follow immediately from the corresponding statements for D. $\square$

Lemma 4.5. *Let $(\tilde{\mathcal{E}}, \tilde{D})$ be an even unbounded Kasparov $A \otimes \mathbb{C}\mathrm{l}_1$-$B$ cycle. Then there exists an odd unbounded Kasparov A-B cycle $(\mathcal{E}, D)$ such that the $A \otimes \mathbb{C}\mathrm{l}_1$-$B$ cycle $(\mathcal{E} \otimes \mathbb{C}^2, D \otimes \gamma^1)$ from Lemma 4.4 represents the same class as $(\tilde{\mathcal{E}}, \tilde{D})$. We call this "halving" an even cycle.*

Proof. The difficulty in this "halving" procedure is that $\tilde{D}$ might not anti-commute with the action of $\mathbb{C}\mathrm{l}_1$ as in the case of a doubled odd cycle. In [Dun16, Thm. 5.1] Dungen shows that $\tilde{D}$ can be modified such that it does, without changing the represented KK-class.

It then follows, using the action of $\mathbb{C}\mathrm{l}_1$, that $\tilde{\mathcal{E}}_0 \cong \tilde{\mathcal{E}}_1$ so assume w.l.o.g. that they are in fact the same space. Take $\mathcal{E} = \tilde{\mathcal{E}}_0$ and D to be $\tilde{D}|_{\mathcal{E}}$. $\square$

Combining Lemmas 4.4 and 4.5 we have a method to move between even $A \otimes \mathbb{C}\mathrm{l}_1$-$B$ cycles and odd A-B cycles that is a bijection at the level of KK-classes. Hence we may describe unbounded representatives of $KK_1(A, B)$ by odd cycles.

Recall that $KK_1(A,B)$ can also be defined as $KK_0(A, B \otimes \mathbb{C}l_1)$. There are "doubling" and "halving" notions related to this picture as well, as in [KS16]. This process of "right-doubling" is analytically actually easier than our "left-doubling" because $\tilde{D}$ is automatically $\mathbb{C}l_1$ linear, however the left-doubling turns out to be much more suitable for our later applications.

4.2 Unbounded Product: Kucerovsky

Similar to Definition 3.19 we have a criterion to decide if a certain unbounded cycle is an unbounded representative for the product of two other unbounded cycles. This result is due to Kucerovsky [Kuc96].

There is one concept in that theorem that we have not introduced, that is the concept of compatible resolvents.

Definition 4.6. *Let D and D_1 be unbounded operators on a Hilbert bimodule $\mathcal{E}$. Then resolvent of D is* compatible *with D_1 if there is a dense submodule $\mathcal{W} \subset \mathcal{E}$ such that $(D + i\lambda)^{-1}(D_1 + i\lambda_1)^{-1}$ maps $\mathcal{W}$ into* $\mathrm{dom}(D_1)$ *for all $\lambda, \lambda_1 \in \mathbb{R} \setminus \{0\}$. The submodule $\mathcal{W}$ is called the* domain of compatibility.

Remark 4.7. *See [Kuc96, Lemma 10] for several sufficient conditions for compatibility.*

Theorem 4.8. *Let $(\mathcal{E}_1, D_1) \in \Psi_0(A,B)$, $(\mathcal{E}_2, D_2) \in \Psi_0(B,C)$. Suppose that $(\mathcal{E}_1 \otimes_B \mathcal{E}_2, D) \in \Psi_0(A,C)$, and furthermore*

- *for all ξ in a dense subset of $A \cdot \mathcal{E}_1$ the (graded) commutator*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix}\right]$$

  is bounded on $\mathrm{dom}(D) \oplus \mathrm{dom}(D_2) \subset (\mathcal{E}_1 \otimes_B \mathcal{E}_2) \oplus \mathcal{E}_2$, *(connection condition)*

- *the resolvent of D is compatible with $D_1 \otimes 1$ or vice versa,*

- *$\langle D_1 \otimes 1\psi, D\psi\rangle + \langle D\psi, D_1 \otimes 1\psi\rangle \geq c\langle\psi,\psi\rangle$ for some $c \in \mathbb{R}$ and ψ in the domain of compatibility. (positivity condition)*

Then $(\mathcal{E}_1 \otimes_B \mathcal{E}_2, D)$ represents the Kasparov product of $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$.

Here ξ is homogeneous and T_ξ is defined as in Definition 3.17.

Proof. See [Kuc96, Thm. 13]. □

Remark 4.9. *We have labelled the first and third conditions with the same names appearing in the definition of the bounded Kasparov product, Definition 3.19, since these properties correspond essentially one-to-one with those.*

4.3 Unbounded Connections

In this section we will provide a cursory introduction of connections. To introduce them properly we would need to include a significant stack of definitions related to finding the "differentiable" elements of C^*-algebras and Hilbert bimodules. We want to avoid this, so we will give simplified definitions and refer to [KL12], [KL13], [Mes14] and [BMS16] for details.

Definition 4.10. *Let $(\mathcal{E}, D)$ be an unbounded Kasparov A-B cycle. Define $\Omega_D(A)$ to be the A-$\mathcal{A}$ bimodule generated by elements of the form $[D, a]$, $a \in \mathcal{A}$.*

Definition 4.11. *Let $(\mathcal{E}_1, D_1)$ be an unbounded Kasparov A-B cycle and $(\mathcal{E}_2, D_2)$ an unbounded Kasparov B-C cycle. We say that ∇ is a* connection *from $(\mathcal{E}_1, D_1)$ to $(\mathcal{E}_2, D_2)$ if ∇ is a map from some dense subset $E \subset \mathcal{E}_1$ to $\mathcal{E}_1 \otimes_B \Omega_{D_2}(B)$, satisfying*

$$\nabla(\psi \cdot b) = \nabla(\psi) \cdot b + \psi \otimes [D_2, b].$$

The connection ∇ is called metric *if*

$$\langle \nabla(\psi), \phi \rangle_{\mathcal{E}_1} + \langle \psi, \nabla(\phi) \rangle_{\mathcal{E}_1} = [D_2, \langle \psi, \phi \rangle_{\mathcal{E}_1}].$$

Note that for this to make sense we need $\langle E, E \rangle_{\mathcal{E}_1} \subset \mathcal{B}$ and $\langle \psi, \phi \otimes \omega \rangle_{\mathcal{E}_1} := \langle \psi, \phi \rangle_{\mathcal{E}_1} \cdot \omega \in \Omega_{D_2}(B)$.

These connections are intended to compute Kasparov products. The Leibniz-rule we saw in Definition 4.11 allows us to define an operator associated to $1 \otimes D_2$ on $\mathcal{E}_1 \otimes_B \mathcal{E}_2$, which is not immediately possible since D_2 does not commute with the B-action.

Definition 4.12. *Let $(\mathcal{E}_1, D_1)$, $(\mathcal{E}_2, D_2)$ and ∇ as in Definition 4.11. Define*

$$\mathrm{dom}(\gamma \otimes_\nabla D_2) := E \otimes_{alg} \mathrm{dom}(D_2)$$

and

$$\begin{aligned} \gamma \otimes_\nabla D_2 : \mathrm{dom}(\gamma \otimes_\nabla D_2) &\to \mathcal{E}_1 \otimes_B \mathcal{E}_2, \\ \psi \otimes \phi &\mapsto \gamma(\psi) \otimes D_2 \phi + \nabla(\gamma(\psi)) \cdot \phi, \end{aligned}$$

where γ is the grading of $\mathcal{E}_1$ and $\xi \otimes [D_2, b] \in \mathcal{E}_1 \otimes \Omega_{D_2}(B)$ acts on $\mathcal{E}_2$ by

$$(\xi \otimes [D_2, b]) \cdot \phi = \xi \otimes [D_2, b]\phi.$$

Theorem 4.13. *Let $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$ be unbounded Kasparov A-B, B-C cycles respectively, and ∇ a connection from $(\mathcal{E}_1, D_1)$ to $(\mathcal{E}_2, D_2)$. Then $(\mathcal{E}_1 \otimes_B \mathcal{E}_2, D_1 \otimes 1 + \gamma \otimes_\nabla D_2)$ represents the product of $(\mathcal{E}_1, D_1)$ and $(\mathcal{E}_2, D_2)$ if $[\gamma \otimes_\nabla D_2, D_1 \otimes 1](D_1 \otimes 1 + i\lambda)^{-1}$ is bounded and $[\gamma \otimes_\nabla D_2, a]$ is bounded for $a \in \mathcal{A}$.*

Remark 4.14. *For the precise statement and proof of this theorem see [KL13].*

The proof of Theorem 4.13 starts by showing that $D_1 \otimes 1 + \gamma \otimes_\nabla D_2$ is indeed self-adjoint and has compact resolvent, which is where the relative bound on the commutator plays an essential role. Then the conditions of Kucerovsky (Theorem 4.8) are checked, one of which we will show here as well.

Lemma 4.15. *Let ∇ be a metric connection from $(\mathcal{E}_1, D_1)$ to $(\mathcal{E}_2, D_2)$, then the connection condition in Theorem 4.8 is satisfied by $D_1 \otimes 1 + \gamma \otimes_\nabla D_2$.*

Proof. This proof is essentially just a computation. We want to show that

$$\begin{aligned}&\left[\begin{pmatrix} D_1 \otimes 1 + 1 \otimes_\nabla D_2 & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix}\right] \\ &\qquad = \begin{pmatrix} 0 & (D_1 \otimes 1 + 1 \otimes_\nabla D_2)T_\xi - (-1)^{\partial\xi} T_\xi D_2 \\ D_2 T_\xi^* - (-1)^{\partial\xi} T_\xi^* (D_1 \otimes 1 + 1 \otimes_\nabla D_2) & 0 \end{pmatrix}\end{aligned}$$

is bounded for ξ in a dense subset of $\mathcal{E}_1$. We will do this for the two components separately.

Let $\phi \in \mathcal{E}_2$, then

$$\begin{aligned}(D_1 \otimes 1 + 1 \otimes_\nabla D_2)T_\xi \phi - (-1)^{\partial\xi} T_\xi D_2 \phi =& D_1\xi \otimes \phi + (-1)^{\partial\xi} \xi \otimes D_2\phi \\ &+ (-1)^{\partial\xi} \nabla(\xi) \cdot \phi - (-1)^{\partial\xi} \xi \otimes D_2\phi, \\ =& D_1\xi \otimes \phi + (-1)^{\partial\xi} \nabla(\xi) \cdot \phi.\end{aligned}$$

While the map $\xi \mapsto \nabla(\xi)$, with $\nabla(\xi)$ interpreted as operator $\mathcal{E}_2 \to \mathcal{E}_1 \otimes_B \mathcal{E}_2$ may be unbounded, the map $\nabla(\xi) : \mathcal{E}_2 \to \mathcal{E}_1 \otimes_B \mathcal{E}_2$ is bounded. Hence this component of the commutator is a bounded map for $\xi \in \mathrm{dom}(D_1 \otimes 1 + \gamma \otimes_\nabla D_2)$.

Let $\psi \otimes \phi \in \mathcal{E}_1 \otimes_B \mathcal{E}_2$, then

$$\begin{aligned}&(D_2 T_\xi^* - (-1)^{\partial\xi} T_\xi^* (D_1 \otimes 1 + 1 \otimes_\nabla D_2))(\psi \otimes \phi) \\ &= D_2(\langle \xi, \psi \rangle_{\mathcal{E}_1} \phi) - (-1)^{\partial\xi} \langle \xi, D_1\psi \rangle_{\mathcal{E}_1} \phi - \langle \xi, \psi \rangle_{\mathcal{E}_1} D_2\phi - \langle \xi, \nabla(\psi) \rangle_{\mathcal{E}_1} \cdot \phi, \\ &= [D_2, \langle \xi, \psi \rangle_{\mathcal{E}_1}]\phi - (-1)^{\partial\xi} \langle D_1\xi, \psi \rangle_{\mathcal{E}_1} \phi - \langle \nabla(\xi), \psi \rangle_{\mathcal{E}_1} \cdot \phi - [D_2, \langle \nabla(\xi), \psi \rangle_{\mathcal{E}_1}]\phi, \\ &= -(-1)^{\partial\xi} \langle D_1\xi, \psi \rangle_{\mathcal{E}_1} \phi - \langle \nabla(\xi), \psi \rangle_{\mathcal{E}_1} \cdot \phi.\end{aligned}$$

Again, this is a bounded map $\mathcal{E}_1 \otimes_B \mathcal{E}_2 \to \mathcal{E}_2$ since all unbounded operators have been moved to ξ, which is fixed. □

4.4 The Canonical Spectral Triple of a Manifold

In this section we construct a specific KK-class associated to a manifold M, called the canonical spectral triple. As we mentioned in Section 4.1, spectral triples are examples of unbounded $C(M)$-$\mathbb{C}$ cycles so this associates a $KK_i(C(M), \mathbb{C})$ class to a manifold. In order to construct this triple we need to define a series of structures on our manifold M. We will cherry-pick the essential definitions in order to streamline the discussion, a full introduction can be found in [Sui15] and [BM89].

Definition 4.16. *Let V be a (finite dimensional) vector space over $\mathcal{F}$ and $Q : V \to \mathcal{F}$ a quadratic form. Define the Clifford algebra $\mathrm{Cl}(V, Q)$ to be the unital $\mathbb{C}$-algebra generated by V subject to the relation*

$$v^2 = Q(v)1.$$

Define a grading on $\mathrm{Cl}(V, Q)$ by setting $\gamma(v_1 \cdot \ldots \cdot v_k) = (-1)^k v_1 \cdot \ldots \cdot v_k$.

Definition 4.17. *Let M be a Riemannian manifold and TM the tangent bundle of M. Let $\{x^\mu\}$ be local coordinates over $U \subset M$. Define the* Clifford bundle *$\mathrm{Cl}(TM)$ locally as the unital algebra bundle generated by $\{\partial_\mu\}$ with*

$$\partial_\mu \partial_\nu + \partial_\nu \partial_\mu = 2g_{\mu\nu}.$$

The transition functions are inherited from the tangent bundle. If $\dim(M) = n$ is even, define the Chirality operator $\gamma = (-i)^{\frac{n}{2}} \partial_1 \partial_2 \cdot \ldots \cdot \partial_n$.

Notation 4.18. *We will usually refer to (local) sections of the Clifford bundle in terms of $\gamma_\mu = \partial_\mu$.*

Let us justify the name "Clifford bundle".

Lemma 4.19. *Let M be a Riemannian manifold with $n = \dim(M)$, then over any point $x \in M$ we have $\mathrm{Cl}(TM)_x \cong \mathrm{Cl}(\mathbb{C}^n, v \mapsto g_x(v, v))$.*

Proof. We only need to prove that the imposed relations imply each other. Suppose $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$, then for $v = v^\mu \gamma_\mu$,

$$v^2 = v^\mu v^\nu \gamma_\mu \gamma_\nu.$$

Our relation allows us to replace the terms $\gamma_\mu \gamma_\nu$ and $\gamma_\nu \gamma_\mu$ together by $2g_{\mu\nu}$, and $\gamma_\nu \gamma_\nu = g_{\nu\nu}$. Using that $g_{\mu\nu} = g_{\nu\mu}$ we get $v^2 = v^\mu v^\nu g_{\mu\nu} = g_x(v, v)$.

For the converse, consider

$$\begin{aligned} (\gamma_\mu + \gamma_\nu)^2 &= \langle \gamma_\mu + \gamma_\nu, \gamma_\mu + \gamma_\nu \rangle, \\ \gamma_\mu^2 + \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu + \gamma_\nu^2 &= \langle \gamma_\mu, \gamma_\mu \rangle + \langle \gamma_\mu, \gamma_\nu \rangle + \langle \gamma_\nu, \gamma_\mu \rangle + \langle \gamma_\nu, \gamma_\nu \rangle, \\ \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= g_{\mu\nu} + g_{\nu\mu}. \end{aligned}$$

Using symmetry of g then completes the argument. □

Definition 4.20. *Let M be a Riemannian manifold. We say that M is* spin^c *if there is a vector bundle $\mathcal{S}$ such that* $\mathbb{C}\mathrm{l}(TM) \cong \mathrm{End}(\mathcal{S})$ *if* $\dim(M)$ *is even and* $\mathbb{C}\mathrm{l}(TM)^0 \cong \mathrm{End}(\mathcal{S})$ *if* $\dim(M)$ *is odd. We call $\mathcal{S}$ a* spinor bundle *and the pair $(M, \mathcal{S})$ a* spin^c structure.

The associated action of $\mathbb{C}\mathrm{l}(TM)$ *on $\mathcal{S}$ is called Clifford action or Clifford multiplication and is denoted $c(\partial_\mu)\psi$ for $\psi \in \mathcal{S}$. There is also an associated action of one-forms on the spinor bundle using the Riemannian metric,* $\mathrm{d}x^\mu$ *acts on $\mathcal{S}$ via the vector $v_\nu = g_{\mu\nu}\mathrm{d}x^\mu$, we denote this action $c(\mathrm{d}x^\mu)\psi$ as well.*

If $\dim(M)$ *is even, the Chirality operator defines a grading on $\mathcal{S}$.*

Let $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ be a continuous inner product on $\mathcal{S}$ and define $L^2(\mathcal{S})$ to be the completion of the compactly supported, continuous bundle maps $C_c(M, \mathcal{S})$ in this inner product.

In order to define the Dirac operator we want to define a connection on the Spinor bundle, which we do by lifting the Levi-Civita connection. We will only do this using local coordinates, although the construction can be done more generally.

Definition 4.21. *Let M be a* spin^c *Riemannian manifold with spinor bundle $\mathcal{S}$. Let $\tilde{\Gamma}^b_{\mu a}$ be the Christoffel symbols for the Levi-Civita connection over $U \subset M$ relative to an orthonormal frame $\{E_a\}$. Since $\mathcal{S}$ is locally $U \times \mathbb{C}^k$ and $\{E_a\}$ is a orthonormal frame we can find $k \times k$ matrices $\{\gamma_a\}$ satisfying*

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}$$

such that the Clifford action of E_a is given by γ_a.

The spin connection $\nabla^{\mathcal{S}}$ *is given in local coordinates by*

$$\nabla^{\mathcal{S}}_{\partial_\mu}\psi = \left(\partial_\mu - \frac{1}{4}\tilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b\right)\psi.$$

Here $\gamma^a = \gamma_a$, the index is raised to use the Einstein summation convention.

Definition 4.22. *Let M be a* spin^c *Riemannian manifold. Using the same notation as in Definition 4.21 we define the* Dirac operator *locally by*

$$D_M\psi = ic(\mathrm{d}x^\mu)\nabla^{\mathcal{S}}_{\partial_\mu}(\psi) = ic(\mathrm{d}x^\mu)\left(\partial_\mu - \frac{1}{4}\tilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b\right)\psi.$$

Theorem 4.23. *Let M be a complete* spin^c *Riemannian manifold, then D_M is essentially self-adjoint on $C_0^\infty(M, \mathcal{S})$ and has compact resolvent.*

Proof. See [GVF01, Thm. 9.15]. □

Definition 4.24. *Let M be a* spin^c *Riemannian manifold. Then the data*

$$(C^\infty(M), L^2(\mathcal{S}), D_M)$$

forms the canonical spectral triple of M.

For the purposes of this thesis we define the fundamental KK-class of a manifold to be the class represented by the canonical spectral triple of that manifold.

Definition 4.25. *Let M be a* spin^c *manifold, the* fundamental class *of M is the class represented by $(L^2(\mathcal{S}_M), D_M) \in KK_{\dim(M)}(C(M), \mathbb{C})$, where $\mathcal{S}_M$ is the spinor bundle and D_M is the Dirac operator. If $\dim(M)$ is even, the grading on $L^2(\mathcal{S}_M)$ is given by the chirality operator. We denote the fundamental class by $[M]$.*

For canonical spectral triples, connections are more familiar.

Lemma 4.26. *Let $(\mathcal{E}, D)$ be an unbounded Kasparov A-$C(M)$ cycle. A metric connection from $(\mathcal{E}, D)$ to $[M]$ may equivalently be given by a map $\nabla : E \to \mathcal{E} \otimes_{C(M)} \Omega_{dR}(M)$, satisfying*

$$\langle \nabla(\psi), \phi\rangle_{\mathcal{E}} + \langle \psi, \nabla(\phi)\rangle_{\mathcal{E}} = \mathrm{d}\langle \psi, \phi\rangle_{\mathcal{E}}$$

where $E \subset \mathcal{E}$ dense and $\Omega_{dR}(M)$ denotes the de-Rham cohomology of M.

Proof. Both $\Omega_{dR}(M)$ and $\Omega_{D_M}(C(M))$ are obtained by applying a differential to $C_0^\infty(M)$, in the case of $\Omega_{dR}(M)$ this is the de-Rham differential, in the case of $\Omega_{D_M}(M)$ this is $f \to [D_M, f]$.

$$\begin{aligned}
[D_M, f] &= ic(\mathrm{d}x^\mu)\nabla^{\mathcal{S}}_{\partial_\mu} f - fic(\mathrm{d}x^\mu)\nabla^{\mathcal{S}}_{\partial_\mu},\\
&= ic(\mathrm{d}x^\mu)\left(\partial_\mu - \frac{1}{4}\tilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b\right) f - fic(\mathrm{d}x^\mu)\left(\partial_\mu - \frac{1}{4}\tilde{\Gamma}^b_{\mu a}\gamma^a\gamma_b\right),\\
&= ic(\mathrm{d}x^\mu)\partial_\mu f - fic(\mathrm{d}x^\mu)\partial_\mu,\\
&= i(\partial_\mu f)c(\mathrm{d}x^\mu).
\end{aligned}$$

Here we use that f commutes with the Clifford multiplication so that we only find the derivative terms. Note that $\mathrm{d}f = (\partial_\mu f) \otimes \mathrm{d}x^\mu$, so there is a clear isomorphism between $\Omega_{dR}(M)$ and $\Omega_{D_M}(C(M))$ sending $\mathrm{d}f \to i(\partial_\mu f)c(\mathrm{d}x^\mu) = [D_M, f]$.

This allows us to formulate connections between $(\mathcal{E}, D)$ and $[M]$ as a map $\nabla : E \to \mathcal{E} \otimes_{C(M)} \Omega_{dR}(M)$, we must however note that $(\psi \otimes \mathrm{d}x^\mu) \cdot \phi = \psi \otimes ic(\mathrm{d}x^\mu)\phi$.

Since $\mathrm{d}\langle \psi, \phi\rangle_{\mathcal{E}}$ maps to $[D_M, \langle \psi, \phi\rangle_{\mathcal{E}}]$ it is immediate that a $\Omega_{dR}(M)$-valued connection satisfying the equation in the statement of this Lemma produces a metric $\Omega_{D_M}(C(M))$-valued connection. □

Remark 4.27. *The appearance of the factor i in this construction is due to the differential nature of connections. Since we want to work with self-adjoint operators and differentials are generally anti-self-adjoint we introduce this factor i.*

The reason we introduce the fundamental class of a manifold is our goal of proving that the shriek module provides a factorization of the fundamental classes, as discussed in the introduction. This is actually a direct consequence of the fact that $[M]$ is also the shriek class associated to the map $M \to \{*\}$ and $(g \circ f)_! = f_! \otimes_{C(Y)} g_!$ for $f : X \to Y$, $g : Y \to Z$ [CS84], but we will not prove this here since we want to work solely in the unbounded framework.

Let us compute the Dirac operator in a simple example, that of $\mathbb{R}^2$, or rather $U = \mathbb{R}^2 \setminus \{(0,0)\}$ since we will work in polar coordinates.

Let us start by showing that $\mathbb{R}^2$ is indeed spin^c. The tangent bundle of $\mathbb{R}^2$ is trivial, $T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ which gives a trivial Clifford bundle as well: $\mathbb{C}\mathrm{l}(T\mathbb{R}^2) = \mathbb{R}^2 \times \mathbb{C}\mathrm{l}(\mathbb{R}^2, v \mapsto \langle v, v \rangle)$.

The Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^2, v \mapsto \langle v, v \rangle)$ is generated by orthonormal v_1 and v_2. Consider the pauli-matrices

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Together with 1 they generate $M_2(\mathbb{C})$ and $v_1 \mapsto \gamma_1$, $v_2 \mapsto \gamma_2$ realizes the required commutation relations of the Clifford algebra. Hence $\mathbb{C}\mathrm{l}(\mathbb{R}^2, v \mapsto \langle v, v \rangle) \cong M_2(\mathbb{C})$. This allows us to find a spinor bundle for $\mathbb{R}^2$ by setting $\mathcal{S} = \mathbb{R}^2 \times \mathbb{C}^2$.

From now on we will work in polar coordinates and over U. We will compute the Levi-Civita connection relative to polar coordinates. Recall that the Christoffel symbols for the Levi-Civita connection are given by

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}).$$

The metric is given by $g = (\mathrm{d}r)^2 + r^2 (\mathrm{d}\theta)^2$, so a straightforward computation shows that

$$\begin{aligned} \Gamma^r_{\theta\theta} &= -r, \\ \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} &= \frac{1}{r}, \end{aligned}$$

while all other Christoffel symbols are zero.

Define an orthonormal frame for $T\mathbb{R}^2|_U = U \times \mathbb{R}^2$ by $e_1 = \partial_r$ and $e_2 = \frac{1}{r}\partial_\theta$. We now compute the Christoffel symbols relative to this orthonormal frame using the

Leibniz-rule for connections.

$$\begin{aligned}
\nabla_{\partial_r}(e_2) &= \nabla_{\partial_r}\left(\frac{1}{r}\partial_\theta\right), \\
&= \left(\frac{\partial}{\partial r}\frac{1}{r}\right)\partial_\theta + \frac{1}{r}\nabla_{\partial_r}(\partial_\theta), \\
&= -\frac{1}{r^2}\partial_\theta + \frac{1}{r}\frac{1}{r}\partial_\theta, \\
&= 0.
\end{aligned}$$

Similar computations for the other symbols yield $\tilde{\Gamma}^j_{ri} = 0$, $\tilde{\Gamma}^2_{\theta 1} = 1$, $\tilde{\Gamma}^1_{\theta 2} = -1$ and $\tilde{\Gamma}^i_{\theta i} = 0$.

Now we can employ Definition 4.21 to find the spin connection $\nabla^{\mathcal{S}}$ for $\mathcal{S} = U \times \mathbb{C}^2$. Choose a basis for $\mathcal{S}$ such that the Clifford action of e_1 is given by γ_1 and the Clifford action of e_2 is given by γ_2. Then

$$\begin{aligned}
\nabla^{\mathcal{S}}_{\partial_r} &= \frac{\partial}{\partial r}, \\
\nabla^{\mathcal{S}}_{\partial_\theta} &= \frac{\partial}{\partial \theta} - \frac{1}{4}(\gamma^1\gamma_2 - \gamma^2\gamma_1), \\
&= \frac{\partial}{\partial \theta} - \frac{1}{2}\gamma^1\gamma^2.
\end{aligned}$$

From here we can compute the Dirac operator by Definition 4.22. We get

$$\begin{aligned}
D_U &= i(c(\mathrm{d}r)\nabla^{\mathcal{S}}_{\partial_r} + c(\mathrm{d}\theta)\nabla^{\mathcal{S}}_{\partial_\theta}), \\
&= i\left(\gamma^1\frac{\partial}{\partial r} + \frac{1}{r}\gamma^2\left(\frac{\partial}{\partial \theta} - \frac{1}{2}\gamma^1\gamma^2\right)\right), \\
&= i\left(\gamma^1\frac{\partial}{\partial r} + \frac{1}{2r}\gamma^1 + \frac{1}{r}\gamma^2\frac{\partial}{\partial \theta}\right).
\end{aligned}$$

The canonical spectral triple for $\mathbb{R}^2$ is then

$$\left(C^\infty(\mathbb{R}^2), L^2(\mathbb{R}^2, \mathbb{C}^2), D_{\mathbb{R}^2}\right).$$

The fundamental class $[\mathbb{R}^2] \in KK_0(C(\mathbb{R}^2), \mathbb{C})$ associated to $\mathbb{R}^2$ is represented by

$$\left(L^2(\mathbb{R}^2, \mathbb{C}^2), D_{\mathbb{R}^2}\right),$$

the grading is given by $\gamma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

In the rest of the text we will write $D_{\mathbb{R}^2}$ for the expression for D_U, this is an abuse of notation but we think it is justified since we only ever use the action of $D_{\mathbb{R}^2}$ away from zero.

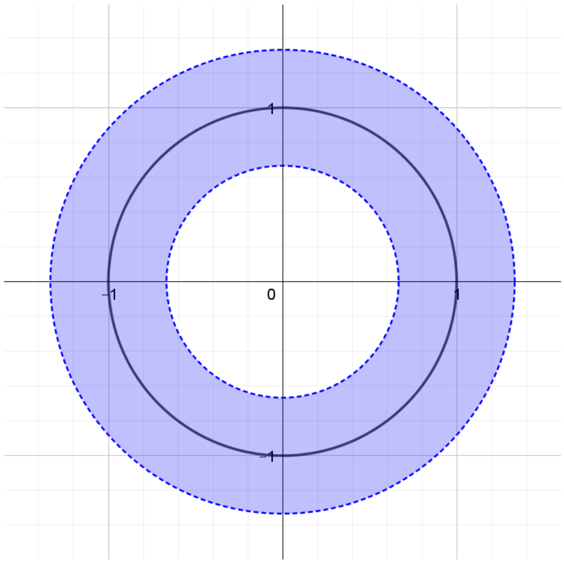

Figure 1: This is a graphical representation of the images of ι and $\tilde{\iota}$. The red line is the image of $\iota : S^1 \hookrightarrow \mathbb{R}^2$ and the blue band is the image of $\tilde{\iota} : S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$.

5 Immersion module

Let us start by introducing some notation. Let $0 < \varepsilon < 1$, we will write $\alpha = \frac{\pi}{2\varepsilon}$ and f will be the function $f(s) = \alpha \tan(\alpha s)$ defined on $(-\varepsilon, \varepsilon)$. Furthermore, let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ and define $\tilde{\iota} : S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$, $\tilde{\iota}(\theta, s) = (\theta, s+1)$ in polar coordinates, see also Figure 1. The notation $\tilde{\iota}$ is inspired by [CS84] who use a similar function on their construction of the shriek module as we will explain in Section 5.2.

We will start our discussion of the shriek module corresponding to ι by introducing the Hilbert $C(S^1)$-$C_0(\mathbb{R}^2)$ bimodule $\mathcal{E}$. Define $\mathcal{E} = C_0(S^1 \times (-\varepsilon, \varepsilon))$ with $C_0(\mathbb{R}^2)$-valued inner product

$$\langle \psi, \phi \rangle_{\mathcal{E}}(\theta, r) := \begin{cases} \frac{1}{r}(\overline{\psi}\phi) \circ \tilde{\iota}^{-1}(\theta, r), & (\theta, r) \in \tilde{\iota}(S^1 \times (-\varepsilon, \varepsilon)), \\ 0, & (\theta, r) \notin \tilde{\iota}(S^1 \times (-\varepsilon, \varepsilon)). \end{cases}$$

The left- and right-actions by $g \in C(S^1)$ and $h \in C_0(\mathbb{R}^2)$ are given by

$$(g \cdot \psi \cdot h)(\theta, s) = g(\theta)\psi(\theta, s)h(\tilde{\iota}(\theta, s)),$$

using polar coordinates for $\mathbb{R}^2$, and we define an operator on $\mathcal{E}$ by

$$(S\psi)(\theta, s) = f(s)\psi(\theta, s) = \frac{\pi}{2\varepsilon} \tan\left(\frac{\pi s}{2\varepsilon}\right) \psi(\theta, s).$$

The goal of this Chapter is to prove that $(\mathcal{E}, S)$ is an unbounded Kasparov $C(S^1)$-$C_0(\mathbb{R}^2)$ cycle that represents the shriek class as constructed in [CS84]. However, we will start by simply showing that $\mathcal{E}$ is indeed a Hilbert bimodule.

Lemma 5.1. *$\mathcal{E}$ as defined above is a Hilbert $C(S^1)$-$C_0(\mathbb{R}^2)$ bimodule.*

Proof. The only property that is not immediate is completeness, we prove this by showing that the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ is equivalent to the sup-norm. Let $\psi \in \mathcal{E}$, then

$$\begin{aligned} ||\psi||_{\mathcal{E}}^2 &= ||\langle \psi, \psi \rangle_{\mathcal{E}}||_{C_0(\mathbb{R}^2)}, \\ &= \sup_{(\theta,r)\in\mathbb{R}^2} \frac{1}{r} |\psi(\theta, r-1)|^2, \\ &= \sup_{(\theta,s)\in S^1\times(-\varepsilon,\varepsilon)} \frac{1}{1+s} |\psi(\theta, s)|^2, \\ &= \left(\sup_{(\theta,s)\in S^1\times(-\varepsilon,\varepsilon)} \frac{1}{\sqrt{1+s}} |\psi(\theta, s)| \right)^2. \end{aligned}$$

Since $\frac{1}{\sqrt{1+s}}$ is bounded between $0 < \frac{1}{\sqrt{1+\varepsilon}} < \frac{1}{\sqrt{1-\varepsilon}}$ on $(-\varepsilon, \varepsilon)$ we get

$$\frac{1}{\sqrt{1+\varepsilon}} ||\psi||_{\sup} \leq ||\psi||_{\mathcal{E}} \leq \frac{1}{\sqrt{1-\varepsilon}} ||\psi||_{\sup}.$$

□

We will also make use of a metric connection on this Hilbert bimodule. We define

$$\begin{aligned} \nabla^{\mathcal{E}} : C_0^\infty(S^1 \times (-\varepsilon, \varepsilon)) &\to \mathcal{E} \otimes \Omega_{dR}(\mathbb{R}^2) \\ \psi &\mapsto \left(\frac{\partial \psi}{\partial s} - \frac{1}{2(s+1)} \psi \right) \otimes \mathrm{d}r + \frac{\partial \psi}{\partial \theta} \otimes \mathrm{d}\theta. \end{aligned}$$

Lemma 5.2. *The map $\nabla^{\mathcal{E}}$ as defined above is a metric connection on $\mathcal{E}$.*

Proof. The connection property is a straightforward check, so we will only show that

$$\langle \nabla^{\mathcal{E}}_{\partial_r}(\psi), \phi \rangle_{\mathcal{E}} + \langle \psi, \nabla^{\mathcal{E}}_{\partial_r}(\phi) \rangle_{\mathcal{E}} = \frac{\partial}{\partial r} (\langle \psi, \phi \rangle_{\mathcal{E}}).$$

The proof for the ∂_θ direction uses the same approach and is simpler.

Since $\tilde{\iota}$ is simply translation in the "radial" direction, we have

$$\frac{\partial}{\partial r} (\psi \circ \tilde{\iota}^{-1}) = \left(\frac{\partial}{\partial s} \psi \right) \circ \tilde{\iota}^{-1}.$$

Assume, without loss of generality, that ψ and ϕ are real valued elements of $C_0^\infty(S^1 \times (-\varepsilon, \varepsilon))$. Then

$$
\begin{aligned}
&\langle \nabla^{\mathcal{E}}_{\partial_r}(\psi), \phi \rangle_{\mathcal{E}} + \langle \psi, \nabla^{\mathcal{E}}_{\partial_r}(\phi) \rangle_{\mathcal{E}} = \\
&\qquad = \frac{1}{r}(\nabla^{\mathcal{E}}_{\partial_r}(\psi)\phi) \circ \tilde{\iota}^{-1} + \frac{1}{r}(\psi \nabla^{\mathcal{E}}_{\partial_r}(\phi)) \circ \tilde{\iota}^{-1}, \\
&\qquad = \frac{1}{r}\left(\frac{\partial \psi}{\partial s}\phi - \frac{1}{2(s+1)}\psi\phi\right) \circ \tilde{\iota}^{-1} + \frac{1}{r}\left(\psi \frac{\partial \phi}{\partial s} - \psi \frac{1}{2(s+1)}\phi\right) \circ \tilde{\iota}^{-1}, \\
&\qquad = \frac{1}{r}\left(\frac{\partial \psi}{\partial s}\phi + \psi \frac{\partial \phi}{\partial s}\right) \circ \tilde{\iota}^{-1} - \frac{1}{r^2}(\psi\phi) \circ \tilde{\iota}^{-1}, \\
&\qquad = \frac{1}{r}\frac{\partial}{\partial r}\left(\psi\phi \circ \tilde{\iota}^{-1}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\right)(\psi\phi) \circ \tilde{\iota}^{-1}, \\
&\qquad = \frac{\partial}{\partial r}\langle \psi, \phi \rangle_{\mathcal{E}}.
\end{aligned}
$$

By Lemma 4.26, $\nabla^{\mathcal{E}}$ is a metric connection relative to $D_{\mathbb{R}^2}$. □

5.1 Analytical Properties

Remark 5.3. *In this section we do not use the explicit function $\alpha \tan(\alpha s)$, so these results hold in greater generality. Specifically, any real-valued continuous function tending to infinity at $\pm\varepsilon$ would work with the same proofs. The motivation for the specific form of $f(s)$ is given in Chapter 6. See also Figure 2.*

We now turn to the analytical properties of S that are required to make $(\mathcal{E}, S)$ into a Kasparov cycle.

Lemma 5.4. *Define $\mathrm{dom}(S) = \{\psi \in \mathcal{E} \,|\, [(\theta, s) \mapsto f(s)\psi(\theta, s)] \in \mathcal{E}\}$. Then $S : \mathrm{dom}(S) \to \mathcal{E}$ is self-adjoint.*

Proof. Suppose $\phi \in \mathrm{dom}(S^*)$, i.e. the functional

$$\eta \mapsto \langle S\eta, \phi \rangle_{\mathcal{E}}$$

is bounded for all $\eta \in \mathrm{dom}(S)$, and let $\xi = S^*\phi \in \mathcal{E}$.

Then $\xi(\theta, s) = f(s)\phi(s, \theta)$ since for all $\psi \in C_c(S^1 \times (-\varepsilon, \varepsilon))$ we have $\psi(\theta, s)\xi(\theta, s) = f(s)\psi(\theta, s)\phi(\theta, s)$. Therefore $S\phi = \xi \in \mathcal{E}$ so that $\phi \in \mathrm{dom}(S)$. Since S is clearly symmetric this proves self-adjointness. □

Lemma 5.5. *The operator $S : \mathrm{dom}(S) \to \mathcal{E}$ defined as in Lemma 5.4 is regular.*

Proof. We need to show that the operator $(1+S^2)^{-1}$ has dense range. This follows by noting that for $\psi \in C_c(S^1 \times (-\varepsilon, \varepsilon))$ also $(1 + S^2)\psi \in C_c(S^1 \times (-\varepsilon, \varepsilon))$, hence the range of $(1 + S^2)^{-1}$ contains $C_c(S^1 \times (-\varepsilon, \varepsilon))$ which is dense. □

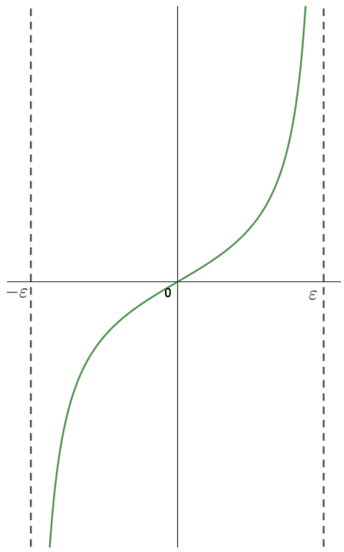

Figure 2: The function $f(s)$. While in Chapter 6 we use explicitly that $f(s) = \frac{\pi}{2\varepsilon}\tan(\frac{\pi s}{2\varepsilon})$, in this chapter it is only relevant that $f(s)$ tends to $\pm\infty$ at $\pm\varepsilon$.

Lemma 5.6. *The operator* $S : \mathrm{dom}(S) \to \mathcal{E}$ *has compact resolvent.*

Proof. We need to show that $(S+i)^{-1} \in \mathcal{K}(\mathcal{E})$, so that it is the limit of a linear combination of rank one operators, as defined in Definition 2.10. Since $f(s)$ tends to $\pm\infty$ as $s \to \pm\varepsilon$ the function $\frac{1}{f(s)+i}$ is in $C_0(S^1 \times (-\varepsilon, \varepsilon))$.

Let $g_n \in C_c((-\varepsilon, \varepsilon))$ be a sequence of compactly supported functions converging to $\frac{1}{f(s)+i}$ in sup-norm and $\chi_n \in C_0((-\varepsilon, \varepsilon))$ such that $\chi_n \equiv 1$ on $\mathrm{supp}(g_n)$. Then $\frac{1}{f(s)+i} = \lim_n |\chi_n\rangle\langle g_n|$. $\square$

We are now ready to state the main result of this section.

Proposition 5.7. *The pair* $(\mathcal{E}, S)$ *is an unbounded* $KK_1(C(S^1), C_0(\mathbb{R}^2))$ *cycle. We call* $(\mathcal{E}, S)$ *the shriek cycle.*

Proof. We have already established that $\mathcal{E}$ is an appropriate bimodule at the start of this Chapter. In Lemma's 5.4, 5.5 and 5.6 we have proven the required analytical properties of S. Therefore $(\mathcal{E}, S)$ is an odd unbounded Kasparov $C(S^1)$-$C_0(\mathbb{R}^2)$ cycle. $\square$

5.2 Homotopy of bounded transform to shriek class

To conclude this section we motivate the name "shriek cycle" for $(\mathcal{E}, S)$ by showing that $(\mathcal{E}, \mathfrak{b}(S))$ is homotopic to the shriek cycle as constructed in [CS84].

Let us first consider the construction before [CS84, Prop. 2.8] for $S^1 \hookrightarrow \mathbb{R}^2$. We choose a map $\tilde{\iota}_{CS} : S^1 \times \mathbb{R} \to \mathbb{R}^2$ which is a diffeomorphism onto a tubular neighbourhood of $\iota(S^1) \subset \mathbb{R}^2$. Then define a $C_0(\mathbb{R}^2)$-valued sesquilinear form $\langle \cdot, \cdot \rangle_{CS}$ on $C_c(S^1 \times \mathbb{R})$ by setting

$$\langle \psi, \phi \rangle_{CS}(x) = \overline{\psi}(\tilde{\iota}_{CS}^{-1}(x))\phi(\tilde{\iota}_{CS}^{-1}(x))$$

for x in the tubular neighbourhood, and $\langle \psi, \phi \rangle_{CS} = 0$ elsewhere. Equip $C_c(S^1 \times \mathbb{R})$ with a left $C(S^1)$ action and a right $C_0(\mathbb{R}^2)$ action by

$$(g \cdot \psi \cdot h)(\theta, s) = g(\theta)\psi(\theta, s)h(\tilde{\iota}_{CS}(\theta, s)).$$

This turns $C_c(S^1 \times \mathbb{R})$ into a pre-Hilbert bimodule, denote by $\mathcal{E}_{CS}$ the corresponding Hilbert $C(S^1)$-$C_0(\mathbb{R}^2)$-bimodule. Let $M : [0, \infty) \to [0, 1]$ be such that $M(0) = 1$ and M has compact support. On $\mathcal{E}_{CS}$ define an operator $F : \mathcal{E}_{CS} \to \mathcal{E}_{CS}$ by

$$(F\psi)(\theta, s) = \sqrt{1 - M(|s|)} \frac{s}{|s|} \psi(\theta, s).$$

In [CS84] there is a Clifford action in this formula, but the Clifford structure in this case is just multiplication by the vector-coordinate.

Choose

$$M(s) = \begin{cases} \frac{1}{1+f(s)^2}, & s \in [0, \varepsilon) \\ 0, & s \geq \varepsilon \end{cases},$$

then

$$(F\psi)(\theta, s) = \begin{cases} -\psi(\theta, s), & s \leq -\varepsilon \\ \frac{f(s)}{\sqrt{1+f(s)^2}} \psi(\theta, s), & s \in (-\varepsilon, \varepsilon) \\ \psi(\theta, s). & s \geq \varepsilon \end{cases}$$

This already bears close resemblance to $(\mathcal{E}, \mathfrak{b}(S))$. The major difference is that $\mathcal{E}$ uses $(-\varepsilon, \varepsilon)$ as fibre with a operator tending to 1 at the edge, while $\mathcal{E}_{CS}$ uses $\mathbb{R}$ as fibre.

We will now define a homotopy between $(\mathcal{E}_{CS}, F)$ and $(\mathcal{E}, \mathfrak{b}(S))$. Choose $\tilde{\iota}_{CS}$ such that $\tilde{\iota}_{CS}|_{S^1 \times (-\varepsilon, \varepsilon)} \equiv \tilde{\iota}$.

Let $R : [0, 1) \to \mathbb{R}$ be any increasing function such that $R(0) = \varepsilon$ and $R(x) \to \infty$ as $x \to 1$. Define $X \subset S^1 \times \mathbb{R} \times [0, 1]$ by $(\theta, s, t) \in X$ if $t = 1$ or $|s| < R(t)$ for

$t < 1$. Set $\mathcal{F} = C_0(X)$, and define a $C_0(\mathbb{R}^2) \otimes C([0,1]) = C_0(\mathbb{R}^2 \times [0,1])$-valued sesquilinear form on $\mathcal{F}$ by

$$\begin{aligned}\langle \cdot, \cdot \rangle_{\mathcal{F}} : C_0(X) \times C_0(X) \to C_0(\mathbb{R}^2 \times [0,1]),\\ \langle \psi, \phi \rangle_{\mathcal{F}}(\theta, r, t) = \overline{\psi}(\tilde{\iota}_{CS}^{-1}(\theta, r), t)\phi(\tilde{\iota}_{CS}^{-1}(\theta, r), t).\end{aligned}$$

Equip $\mathcal{F}$ with a left-$C(S^1)$ and right-$C_0(\mathbb{R}^2 \times [0,1])$ action by

$$(f \cdot \psi \cdot g)(\theta, s, t) = f(\theta)\psi(\theta, s, t)g(\tilde{\iota}_{CS}(\theta, s), t).$$

Note that the norm on $\mathcal{F}$ induced by this inner product is simply the sup-norm on $C_0(X)$, so that $\mathcal{F}$ is indeed a Hilbert bimodule. Then $\mathcal{L}(\mathcal{F}) = C_b(X)$ and $\mathcal{K}(\mathcal{F}) = C_0(X)$.

Now we define an operator G on $\mathcal{F}$ by

$$(G\psi)(\theta, s, t) = \begin{cases} -\psi(\theta, s, t), & s \leq -\varepsilon \\ \frac{f(s)}{\sqrt{1+f(s)^2}}\psi(\theta, s, t), & s \in (-\varepsilon, \varepsilon) \\ \psi(\theta, s, t). & s \geq \varepsilon \end{cases}$$

Note that $G^2 - 1$ is in $\mathcal{K}(\mathcal{F})$ since it is in $C_0(X)$.

Let B_i be the $C_0(\mathbb{R}^2 \times [0,1])$-$C_0(\mathbb{R}^2)$ Hilbert bimodule corresponding to $\Phi_i : C_0(\mathbb{R}^2 \times [0,1]) \to C_0(\mathbb{R}^2)$, $(\Phi_i f)(\theta, r) = f(\theta, r, i)$ by Example 2.15, for $i = 0, 1$.

Lemma 5.8. *$(\mathcal{F} \otimes_{C_0(\mathbb{R}^2 \times [0,1])} B_1, G \otimes 1)$ is unitarily equivalent to $(\mathcal{E}_{CS}, F)$.*

Proof. We will use the map

$$\begin{aligned}U : \mathcal{F} \otimes_{C_0(\mathbb{R}^2 \times [0,1])} B_1 \to \mathcal{E}_{CS}\\ U(\psi \otimes g)(\theta, s) = \psi(\theta, s, 1)g(\tilde{\iota}_{CS}(\theta, s)).\end{aligned}$$

Let us first check that this is unitary, it is sufficient to check this on elementary tensors. Write $(\theta, r) = \tilde{\iota}_{CS}(\theta, s)$.

$$\begin{aligned}\langle U(\psi \otimes f), U(\phi \otimes g)\rangle_{CS}(\theta, r) &= \overline{U(\psi \otimes f)}(\theta, s)U(\phi \otimes g)(\theta, s),\\ &= \overline{\psi(\theta, s, 1)f(\theta, r)}\phi(\theta, s, 1)g(\theta, r),\\ &= \langle f, \langle \psi, \phi\rangle_{\mathcal{F}} \cdot g\rangle_{C_0(\mathbb{R}^2)}(\theta, r),\\ &= \langle \psi \otimes f, \phi \otimes g\rangle_{C_0(\mathbb{R}^2)}(\theta, r).\end{aligned}$$

Outside of the range of $\tilde{\iota}_{CS}$ both sides are 0, so U is a unitary. Moreover U is surjective, since it is straightforward to show that $C_c(S^1 \times \mathbb{R})$ lies within the image of U.

It remains to show that $U(G\otimes 1) = FU$. This is again a computation. Let $s \in (-\varepsilon, \varepsilon)$, then

$$
\begin{aligned}
(U(G\otimes 1)(\psi\otimes g))(\theta,s) &= (U(G\psi\otimes g))(\theta,s),\\
&= (G\psi)(\theta,s,1)g(\theta,r),\\
&= \frac{f(s)}{\sqrt{1+f(s)^2}}\psi(\theta,s,1)g(\theta,r),\\
&= FU(\psi\otimes g)(\theta,s).
\end{aligned}
$$

Outside $(-\varepsilon,\varepsilon)$ this still holds by the same argument, except we take ± 1 instead of $\frac{f(s)}{\sqrt{1+f(s)^2}}$. □

Lemma 5.9. *$(\mathcal{F}\otimes_{C_0(\mathbb{R}^2\times[0,1])} B_0, G\otimes 1)$ is unitarily equivalent to $(\mathcal{E}, \mathfrak{b}(S))$.*

Proof. This time we use the map

$$
\begin{aligned}
&V : \mathcal{F}\otimes_{C_0(\mathbb{R}^2\times[0,1])} B_0 \to \mathcal{E}\\
&V(\psi\otimes g)(\theta,s) = \sqrt{s+1}\,\psi(\theta,s,0)g(\tilde{\iota}(\theta,s)).
\end{aligned}
$$

The proof that this is a unitary map intertwining $G\otimes 1$ and $\mathfrak{b}(S)$ proceeds exactly like the previous Lemma, the factor $\sqrt{s+1}$ cancels out the $\frac{1}{r}$ appearing in $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. □

Lemmas 5.8 and 5.9 show that $(\mathcal{F}, G)$ is a homotopy between $(\mathcal{E}, \mathfrak{b}(S))$ and $(\mathcal{E}_{CS}, F)$ which justifies the name shriek cycle for $(\mathcal{E}, S)$. In principle this also proves that $[S^1] = (\mathcal{E}, S)\otimes_{C_0(\mathbb{R}^2)} [\mathbb{R}^2]$ as KK-cycles, since it holds at the bounded level. However, we want to prove this purely in the unbounded setting.

An important motivation for using the unbounded picture already appears in the proof of Lemma 5.9, where we see that the $\frac{1}{r}$ factor in $\langle\cdot,\cdot\rangle_{\mathcal{E}}$, which is supposed to account for the mean curvature of S^1 in $\mathbb{R}^2$, is irrelevant at the bounded level. At the unbounded level, this factor of $\frac{1}{r}$ influences which connections are metric, thereby influencing the form of the product operator. Proving the factorization for our toy model $S^1 \hookrightarrow \mathbb{R}^2$ at the unbounded level paves the way for more interesting examples where curvature terms may appear in the factorization, as in [KS16].

6 Index Class

In [Ech17] S. Echterhoff shows that the definition of $KK_1(A, B)$ using Clifford algebras as in Definition 3.13 is equivalent to a definition using suspensions instead, which is more usual in K-theory. To this end Echterhoff constructs a "Dirac element" $\alpha \in KK_0(C_0(\mathbb{R}) \otimes \mathbb{C}\mathrm{l}_1, \mathbb{C})$ and a "dual Dirac element" $\beta \in KK_0(\mathbb{C}, C_0(\mathbb{R}) \otimes \mathbb{C}\mathrm{l}_1)$, for which he shows $\alpha \otimes_{\mathbb{C}} \beta = \mathbb{1}_{C_0(\mathbb{R}) \otimes \mathbb{C}\mathrm{l}_1}$ and $\beta \otimes_{C_0(\mathbb{R}) \otimes \mathbb{C}\mathrm{l}_1} \alpha = \mathbb{1}_{\mathbb{C}}$. The operator in the Dirac element is $\frac{\mathrm{d}/\mathrm{d}x}{\sqrt{1+(\mathrm{d}/\mathrm{d}x)^2}}$ which we recognize as $\mathfrak{b}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$, while the operator in the dual Dirac element is $\frac{x}{\sqrt{1+x^2}}$ which we can recognize as $\mathfrak{b}(x)$.

In this section we prove a result which is similar in spirit to Echterhoff's dual Dirac. We take the radial part of our unbounded operator S from the shriek module in Section 5 and combine it with a derivative in such a way that we find the unit in $KK_0(\mathbb{C}, \mathbb{C})$.

Proposition 6.1. *The triple*

$$\left(L^2((-\varepsilon, \varepsilon), \mathbb{C}^2), T = \begin{pmatrix} 0 & i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) s$$

with $f(s) = \alpha \tan(\alpha s)$ is an unbounded cycle in $KK_0(\mathbb{C}, \mathbb{C})$ that represents the unit in $KK_0(\mathbb{C}, \mathbb{C})$. The set $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ is a core for T. We write $\mathbb{1}$ for the class in $KK_0(\mathbb{C}, \mathbb{C})$ represented by this triple.

Proof. The left- and right actions of $\mathbb{C}$ on $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ are simply scalar multiplication and the $\mathbb{C}$-valued inner product on $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ is the standard L^2 inner product. This makes $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ a Hilbert $\mathbb{C}$-$\mathbb{C}$ bimodule.

In Proposition 6.2 we will show that the operator T_0 defined on $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ by the same matrix as T is essentially self-adjoint, so that $T = \overline{T_0}$ is self-adjoint. Then in Proposition 6.11 we will show that T has compact resolvent. Therefore our data defines a $KK_0(\mathbb{C}, \mathbb{C})$ cycle.

Finally in Proposition 6.16 we show that $\mathrm{Index}(\mathfrak{b}(T)) = 1$ so that the cycle indeed represent the unit in $KK_0(\mathbb{C}, \mathbb{C})$. $\square$

6.1 Self-adjointness

Proposition 6.2. *The operator $T_0 : C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2) \to L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ given by*

$$T_0 = \begin{pmatrix} 0 & i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix}$$

is essentially self-adjoint.

Before proving this we give an example of the method of proof we are after, which is taken from [Lax02, Ch. 33].

Example 6.3. *Consider the operator $d = i\frac{\mathrm{d}}{\mathrm{d}x} : C_c^\infty(\mathbb{R}) \to L^2(\mathbb{R})$. We will show that d is essentially self-adjoint by showing that the range of $d \pm i$ is dense in $L^2(\mathbb{R})$, and then invoking Theorem 2.32.*

Suppose $g = (d + \lambda i)u$ for $u \in C_c^\infty(\mathbb{R})$, i.e. u satisfies the differential equation

$$g = iu' + \lambda iu.$$

We want to find a pair of integrating factors for this equation, that means two functions $I, J : \mathbb{R} \to \mathbb{C}$ such that $Jg = \frac{\mathrm{d}}{\mathrm{d}x}(Iu)$.

Since $g = (d + \lambda i)u$ we get a differential equation for I and J by

$$\begin{aligned} Jg &= \frac{\mathrm{d}}{\mathrm{d}x}(Iu), \\ J(iu' + \lambda iu) &= I'u + Iu', \end{aligned}$$

for all $u \in C_c^\infty(\mathbb{R})$, so $iJ = I$ and $I' = i\lambda J = \lambda I$. We choose I such that $I(0) = 1$, so $I(x) = e^{\lambda x}$ and consequently $J(x) = -ie^{\lambda x}$.

Then

$$\int_{\mathbb{R}} J(x)g(x)\mathrm{d}x = \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x}(Iu)(x)\mathrm{d}x = 0,$$

since u has compact support. Hence if $g \in \operatorname{ran}(d + \lambda i)$ then $\int_{\mathbb{R}} Jg = 0$, so we have an "orthogonality" condition for functions in the range of $d + \lambda i$. Furthermore, g has compact support and is smooth.

We also have a converse, suppose $g \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} Jg = 0$. Define

$$u(x) = I(x)^{-1} \int_{-\infty}^{x} J(y)g(y)\mathrm{d}y.$$

Then u is smooth and has compact support since $\int_{\mathbb{R}} Jg = 0$. Furthermore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(Iu)(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{x} J(y)g(y)\mathrm{d}y, \\ I'(x)u(x) + I(x)u'(x) &= J(x)g(x), \\ I(x)(u'(x) + \lambda u(x)) &= -iI(x)g(x), \\ (d + \lambda i)u(x) &= g(x). \end{aligned}$$

So we have established that

$$\operatorname{ran}(d+\lambda i) = \left\{ g \in C_c^\infty(\mathbb{R}) \,\middle|\, \int_{\mathbb{R}} J(x)g(x)\mathrm{d}x = 0 \right\}$$

where $J(x) = -ie^{-\lambda x}$.

For $\lambda \neq 0$ *we have* $J \notin L^2(\mathbb{R})$, *so we can use Lemma 6.4 to conclude that the range of* $d \pm i$ *is dense. Symmetry of* d *follows from integration by parts, so Theorem 2.32 proves that* d *is essentially self-adjoint on the domain* $C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$.

Lemma 6.4. *Let* $\Omega \subset \mathbb{R}^d$ *and* $j \in C(\Omega, \mathbb{C}^n)$, $j \notin L^2(\Omega, \mathbb{C}^n)$. *Then*

$$K_j = \left\{ g \in C_c^\infty(\Omega) \middle| \int_\Omega \langle j(x), g(x)\rangle \mathrm{d}x = 0 \right\}$$

is dense.

Proof. Define a linear functional $\langle j| : C_c^\infty(\Omega) \to \mathbb{C}$ by $\langle j|f = \int_\Omega \langle j(x), f(x)\rangle \mathrm{d}x$. Our first step is to prove that $\langle j|$ is unbounded.

Suppose $\langle j|$ is bounded on $C_c^\infty(\Omega, \mathbb{C}^n)$ with respect to the $L^2(\Omega, \mathbb{C}^n)$-norm. Then $\langle j|$ extends to a bounded linear functional on $L^2(\Omega, \mathbb{C}^n)$, given by $\psi \mapsto \langle \tilde{j}, \psi\rangle$ for some $\tilde{j} \in L^2(\Omega, \mathbb{C}^n)$ by Riesz-representation.

But then $\int_\Omega \langle j(x), g(x)\rangle \mathrm{d}x = \int_\Omega \langle \tilde{j}(x), g(x)\rangle$ for all $g \in C_c^\infty(\Omega)$, which implies $j(x) = \tilde{j}(x)$. This is in contradiction with our assumption that $j \notin L^2(\Omega, \mathbb{C}^n)$.

So $\langle j|$ is an unbounded linear functional on $C_c^\infty(\Omega, \mathbb{C}^n)$. Therefore there exists a sequence $(\delta_m)_{m\in\mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{C}^n)$ such that $\langle j|\delta_m = 1$ and $||\delta_m||_{L^2} < \frac{1}{m}$ for all $m \in \mathbb{N}$.

Let $\psi \in L^2(\Omega, \mathbb{C}^n)$ and $\epsilon > 0$ be arbitrary. Then there is an $\psi_1 \in C_c^\infty(\Omega, \mathbb{C}^n)$ such that $||\psi - \psi_1||_{L^2} < \frac{1}{2}\epsilon$. Define $\alpha = \langle j|\psi_1$ and find M such that $\frac{\alpha}{M} < \frac{1}{2}\epsilon$. Set $\psi_2 = \psi_1 - \alpha\delta_M$, then $||\psi - \psi_2||_{L^2} < \varepsilon$ and $\langle j|\psi_2 = 0$, proving density of K_j. $\square$

We will now apply the same method to Proposition 6.2, we first find analogues of I and J, show that $\operatorname{ran}(T_0 + \lambda i)$ is the "orthogonal complement" of J and then show that this is dense.

Lemma 6.5. *Suppose* $u, g \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ *and* $\lambda^2 = \alpha^2$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}x} I_\lambda u = J_\lambda g$$

if and only if $g = (T_0 + \lambda i)u$, *for*

$$\begin{aligned}
I_\lambda(s) &= \begin{pmatrix} 1 + sf(s) & \lambda s \\ \frac{1}{\lambda} f(s) & 1 \end{pmatrix}, \\
J_\lambda(s) &= -i \begin{pmatrix} \lambda s & 1 + sf(s) \\ 1 & \frac{1}{\lambda} f(s) \end{pmatrix}.
\end{aligned}$$

Proof. We will show that

$$J_\lambda^{-1} \frac{\mathrm{d}}{\mathrm{d}x} I_\lambda = (T_0 + \lambda i)$$

which proves the Lemma.

First we use the Leibniz identity, so

$$\begin{aligned}
J_\lambda^{-1} \frac{\mathrm{d}}{\mathrm{d}x} I_\lambda &= J_\lambda^{-1} I_\lambda \frac{\mathrm{d}}{\mathrm{d}x} + J_\lambda \left(\frac{\mathrm{d}}{\mathrm{d}x} I_\lambda \right), \\
&= J_\lambda^{-1} I_\lambda \frac{\mathrm{d}}{\mathrm{d}x} + J_\lambda^{-1} I_\lambda'.
\end{aligned}$$

Let us compute these matrices

$$\begin{aligned}
J_\lambda^{-1} I_\lambda &= -i \begin{pmatrix} \frac{1}{\lambda} f(s) & -1 - sf(s) \\ -1 & \lambda s \end{pmatrix} \begin{pmatrix} 1 + sf(s) & \lambda s \\ \frac{1}{\lambda} f(s) & 1 \end{pmatrix}, \\
&= -i \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
J_\lambda^{-1} I_\lambda' &= -i \begin{pmatrix} \frac{1}{\lambda} f(s) & -1 - sf(s) \\ -1 & \lambda s \end{pmatrix} \begin{pmatrix} f(s) + sf'(s) & \lambda \\ \frac{1}{\lambda} f'(s) & 0 \end{pmatrix}, \\
&= -i \begin{pmatrix} \frac{1}{\lambda}(f(s)^2 - f'(s)) & f(s) \\ -f(s) & -\lambda \end{pmatrix}.
\end{aligned}$$

We now use that for $f(s) = \alpha \tan(\alpha s)$ we have

$$\begin{aligned}
f(s)^2 - f'(s) &= \alpha^2 \tan(\alpha s)^2 - \alpha^2 \frac{1}{\cos(\alpha s)^2}, \\
&= \alpha^2 \left(\frac{\sin(\alpha s)^2}{\cos(\alpha s)^2} - \frac{1}{\cos(\alpha s)^2} \right), \\
&= -\alpha^2.
\end{aligned}$$

Therefore, since $\alpha^2 = \lambda^2$,

$$J_\lambda^{-1} I_\lambda' = -i \begin{pmatrix} -\lambda & f(s) \\ -f(s) & -\lambda \end{pmatrix},$$

which proves

$$J_\lambda^{-1} \frac{\mathrm{d}}{\mathrm{d}x} I_\lambda = J_\lambda^{-1} I_\lambda \frac{\mathrm{d}}{\mathrm{d}x} + J_\lambda^{-1} I_\lambda' = T_0 + \lambda i.$$

□

Remark 6.6. *We have similar functions for $\lambda^2 \neq \alpha^2$, define $g(s) = \frac{1}{2} e^{\sqrt{\lambda^2-\alpha^2}s} + \frac{1}{2} e^{-\sqrt{\lambda^2-\alpha^2}s}$. Then*

$$I_\lambda(s) = \begin{pmatrix} g(s) + \frac{1}{\lambda^2-\alpha^2} f(s) g'(s) & \frac{\lambda}{\lambda^2-\alpha^2} g'(s) \\ \frac{1}{\lambda}(g'(s) + f(s)g(s)) & g(s) \end{pmatrix},$$

and $J_\lambda(s) = -i I_\lambda(s) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We restrict to the case $\lambda^2 = \alpha^2$ since it is sufficient and provides a simpler expression.

The next step, following Example 6.3, is to show that the range of $T_0 + \lambda i$ is the "orthogonal complement" of J_λ in $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C})$.

Lemma 6.7. *For $\lambda = \pm\alpha$ and J_λ as in Lemma 6.5 we have*

$$\operatorname{ran}(T_0 + \lambda i) = \left\{ g \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2) \,\middle|\, \int_{-\varepsilon}^{\varepsilon} J(x) g(x) \mathrm{d}x = 0 \right\}.$$

Proof. Suppose $g = (T_0 + \lambda i)u$, then by Lemma 6.5

$$\int_{-\varepsilon}^{\varepsilon} J(x) g(x) \mathrm{d}x = \int_{-\varepsilon}^{\varepsilon} \left(\frac{\mathrm{d}}{\mathrm{d}x} I(x) u(x) \right) \mathrm{d}x = 0,$$

since $u \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$. Also, g is indeed in $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$.

For the converse, suppose $g \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ such that $\int Jg = 0$. Define

$$u(x) = I^{-1}(x) \int_{-\varepsilon}^{x} J(y) g(y) \mathrm{d}y,$$

then certainly $u \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} I(x) u(x) = J(x) g(x).$$

Then by Lemma 6.5 we have $(T_0 + \lambda i)u = g$. □

Finally we want to show that the range of $T_0 + \lambda i$ is dense for $\lambda = \pm\alpha$, this is very similar to Lemma 6.4 but not entirely, since this time we are dealing with a matrix valued function.

Lemma 6.8. *The range of $T_0 + \lambda i$ is dense for $\lambda = \pm\alpha$.*

Proof. Write j_1 and j_2 for the rows of J_λ, so

$$j_1(s) = -i\begin{pmatrix} \lambda s \\ 1 + sf(s) \end{pmatrix},$$
$$j_2(s) = -i\begin{pmatrix} 1 \\ \frac{1}{\lambda}f(s) \end{pmatrix}.$$

Then Lemma 6.7 tells us that the range of $T_0 + \lambda i$ is $K_{j_1} \cap K_{j_2}$, in the notation of Lemma 6.4.

To prove density of $K_{j_1} \cap K_{j_2}$ we use the same strategy as in Lemma 6.4 to obtain two sequences $(\delta^1_m)_{m\in\mathbb{N}}$ and $(\delta^2_m)_{m\in\mathbb{N}}$ such that $\langle j_i | \delta^i_m = 1$ and $||\delta^i_m||_{L^2} < \frac{1}{m}$.

Write $\delta^i_{m,1}$ and $\delta^i_{m,2}$ for the first and second components of δ^i_m respectively, and $\tau : (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$, $\tau(x) = -x$. Since the first component of j_1 is odd, while the second component is even we may replace δ^1_m by

$$\tilde{\delta}^1_m = \frac{1}{2}\begin{pmatrix} \delta^1_m - \delta^1_m \circ \tau \\ \delta^1_m + \delta^1_m \circ \tau \end{pmatrix}.$$

On the other hand, the first component of j_2 is even, while the second component is odd, so we may replace δ^2_m by

$$\tilde{\delta}^2_m = \frac{1}{2}\begin{pmatrix} \delta^2_m + \delta^2_m \circ \tau \\ \delta^2_m - \delta^2_m \circ \tau \end{pmatrix}.$$

Replacing δ^i_m by $\tilde{\delta}^i_m$ does not change the values of $\langle j_i | \delta^i_m$, and it does not increase the norm of the δ^i_m. Furthermore, since the corresponding components of j_i and $\tilde{\delta}^i_m$ now have opposite parity $\langle j_1 | \tilde{\delta}^2_m = \langle j_2 | \tilde{\delta}^1_m = 0$.

We can now complete the density proof similar to the final step in Lemma 6.4. Let $\psi \in L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ and $\epsilon > 0$ be arbitrary. Then there is a $\psi_1 \in C^\infty_c((-\varepsilon, \varepsilon), \mathbb{C}^2)$ such that $||\psi - \psi_1||_{L^2} < \frac{1}{3}\epsilon$. Let $\alpha_i = \langle j_i | \psi_1$ for $i = 1, 2$ and find M such that $\frac{\alpha_i}{M} < \frac{1}{3}\epsilon$. Then $\psi_2 = \psi_1 - \alpha_1\tilde{\delta}^1_M - \alpha_2\tilde{\delta}^2_M$ satisfies both $\langle j_1 | \psi_2 = 0$, $\langle j_2 | \psi_2 = 0$ and $||\psi - \psi_2||_{L^2} < \epsilon$. ☐

We are now ready to prove Proposition 6.2.

Proof. [Proof of Proposition 6.2]

The operator T_0 is clearly symmetric and by Lemma 6.8 the range of $T_0 \pm \alpha i$ is dense. Then by Theorem 2.32 T_0 is essentially self-adjoint. $\square$

Define T to be the closure of T_0, which is then self-adjoint.

6.2 Compact Resolvent

The other property of T we need is the compact resolvent, the main tool we use is a computation of $T^2 + \lambda^2 = (T + i\lambda)(T - i\lambda)$. That computation allows us to prove that the Sobolev norm of ψ gives a lower bound for the graph-norm of ψ corresponding to $T + \lambda i$.

Lemma 6.9. *The graph-norm of $T \pm \lambda i$ is larger than the Sobolev norm. Indeed, for $\psi \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ we have $||\psi||^2 + ||(T + i\lambda)\psi||^2 > ||\psi||^2 + ||\psi'||^2$.*

Proof. We want to compute $||(T + i\lambda)\psi||^2$ for $\psi \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$, the domain of T_0. The claim then follows for T by continuity. Using the symmetry of T this equals $\langle \psi, (T^2 + \lambda^2)\psi \rangle$, so let us compute T^2.

$$T^2 = \begin{pmatrix} 0 & i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix}^2 = \begin{pmatrix} -\partial_s^2 - f'(s) + f(s)^2 & 0 \\ 0 & -\partial_s^2 + f'(s) + f(s)^2 \end{pmatrix}.$$

Therefore

$$\langle \psi, (T^2 + \lambda^2)\psi \rangle = \langle \psi, -\psi'' \rangle + \langle \psi, \begin{pmatrix} f(s)^2 - f'(s) + \lambda^2 & 0 \\ 0 & f(s)^2 + f'(s) + \lambda^2 \end{pmatrix} \psi \rangle.$$

For $\lambda^2 \geq \alpha^2$ both $f(s)^2 \pm f'(s) + \lambda^2 \geq 0$, so the second term on the right-hand-side is positive. Hence

$$\langle \psi, (T^2 + \lambda^2)\psi \rangle \geq \langle \psi, -\psi'' \rangle.$$

By partial integration $\langle \psi, -\psi'' \rangle = \langle \psi', \psi' \rangle$ so we find that

$$||(T + \lambda i)\psi||^2 \geq ||\psi'||^2.$$

This proves the desired result. $\square$

Corollary 6.10. *The domain of T is contained in the first-order Sobolev space $H^1((-\varepsilon, \varepsilon), \mathbb{C}^2)$.*

Proposition 6.11. *The resolvent $(T + \lambda i)^{-1}$ is compact for $\lambda = \pm\alpha$ and hence for all $\lambda \in \rho(T)$.*

Proof. Define $D = \{\psi \in L^2((-\varepsilon, \varepsilon), \mathbb{C}^2) | \, ||\psi|| \leq 1\}$ the unit ball in L^2. We will prove that $M := (T + \lambda i)^{-1} D$ is pre-compact.

Let $\psi = (T + \lambda i)^{-1}\phi$, $\phi \in D$. Then $||\psi|| \leq |\lambda|^{-1}$ since $||(T + \lambda i)^{-1}|| \leq |\lambda|^{-1}$ and $\psi \in \mathrm{dom}(T) \subset H^1((-\varepsilon, \varepsilon), \mathbb{C}^2)$ by Corollary 6.10. Furthermore Lemma 6.9 tells us that

$$||\psi'|| \leq ||(T + \lambda i)\psi|| = ||\phi|| \leq 1.$$

Since $\alpha < 1$ by construction we may conclude that

$$M \subset \left\{\psi \in H^1((-\varepsilon, \varepsilon), \mathbb{C}^2) \middle| \, ||\psi'|| \leq \alpha^{-1}, ||\psi|| \leq \alpha^{-1}\right\}.$$

By the Rellich embedding theorem the right hand side set is compact, so that M is pre-compact. Compactness of the resolvents for $\lambda \neq \pm\alpha$ follows from the first resolvent identity $(T + \lambda i)^{-1} = (T + \alpha i)^{-1} + (\lambda - \alpha)(T + \lambda i)^{-1}(T + \alpha i)^{-1}$. $\square$

Remark 6.12. *The operator $T^2 + \lambda^2$ is actually a Schrödinger type operator on $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$, which for λ large enough has positive potential. It is a classical result that Schrödinger operators with bounded potential on a bounded domain and Schrödinger operators on an unbounded domain with a confining potential have compact resolvents. The reason we did not use these classical results is that we are dealing with a combined case here, while $f(s)^2 - f'(s) + \lambda^2$ is a bounded potential, $f(s)^2 + f'(s) + \lambda^2$ is unbounded (it is, however, confining). Therefore we chose to do a direct proof along the lines of proofs for Schrödinger operators as found in [RS80].*

Remark 6.13. *An alternative proof goes as follows. Note that $f(s)^2 - f'(s) + \alpha^2 = 0$ so that the upper left component of $T^2 + \alpha^2$ is simply $-\partial_s^2$ which clearly has compact resolvent, let ψ_n denote the corresponding basis of eigenvectors. Then use the transformation $\psi_n \mapsto \phi_n = \frac{1}{\alpha}(\psi_n' + f\psi)$ to construct eigenvectors for the bottom right component of $T^2 + \alpha^2$ with the same eigenvalues.*

The vectors (ψ_n, ϕ_n) then form a set of orthonormal eigenvectors for $T^2 + \alpha^2$ with eigenvalues tending to infinity, so that $(T^2 + \alpha^2)^{-1}$ is compact. From there we can construct a set of eigenvectors for T that also have eigenvalues tending to infinity proving that T has compact resolvent.

We did not use this method since it relies heavily on the precise choice for f and α, while the proof in Proposition 6.11 relies only on $f(s)^2 - f'(s)$ being bounded from below, which is a much lighter condition.

6.3 Multiplicative Unit

We will now show that $(L^2((-\varepsilon,\varepsilon),\mathbb{C}^2),\mathfrak{b}(T))$ has index 1, so that this cycle represents the multiplicative unit in $KK_0(\mathbb{C},\mathbb{C})$ by Section 3.3.

Lemma 6.14. *Let H be a Hilbert space and $D:\mathrm{dom}(D)\to H$ an unbounded operator. Then*

$$\dim\ker D = \dim\ker\mathfrak{b}(D).$$

Proof. Recall from Definition 2.30 that

$$\mathfrak{b}(D) = D(1+D^*D)^{-\frac{1}{2}}.$$

On $\mathrm{dom}(D)$ this equals $(1+DD^*)^{-\frac{1}{2}}D$, so if $D\psi=0$ also $\mathfrak{b}(D)\psi=0$.

Conversely, we have

$$D = \mathfrak{b}(D)(1-\mathfrak{b}(D)^*\mathfrak{b}(D))^{-\frac{1}{2}},$$

defined on $\mathrm{dom}(D)=\mathrm{ran}\left((1-\mathfrak{b}(D)^*\mathfrak{b}(D))^{\frac{1}{2}}\right)$. Suppose $\mathfrak{b}(D)\psi=0$, then $(1+\mathfrak{b}(D)^*\mathfrak{b}(D))^{\frac{1}{2}}\psi=\psi$ so $\psi\in\mathrm{dom}(D)$. On $\mathrm{dom}(D)$ we may move $\mathfrak{b}(D)$ to the right so that $D\psi=(1-\mathfrak{b}(D)\mathfrak{b}(D)^*)^{-\frac{1}{2}}\mathfrak{b}(D)$, so if $\mathfrak{b}(D)\psi=0$, also $D\psi=0$.

Therefore $\ker(D)=\ker(\mathfrak{b}(D))$. ∎

Remark 6.15. *A more complete treatment of the bounded transform can be found in [Lan95, Ch. 10].*

Proposition 6.16. $\mathrm{Index}(\mathfrak{b}(T))=1$

Proof. Write $T_+=i\partial_s+if(s)$ and $T_-=i\partial_s-if(s)=T_+^*$ so that $T=\begin{pmatrix}0 & T_+\\ T_- & 0\end{pmatrix}$. The bounded transform of T is

$$\mathfrak{b}(T)=\begin{pmatrix}0 & T_-(1+T_+T_-)^{-\frac{1}{2}}\\ T_+(1+T_+T_-)^{-\frac{1}{2}} & 0\end{pmatrix}.$$

so that the (graded) index of T is the index of $\mathfrak{b}(T_+)=T_+(1+T_+T_-)^{-\frac{1}{2}}$.

By Lemma 6.14 $\ker\mathfrak{b}(T_+)=\ker T_+$ and $\ker\mathfrak{b}(T_+)^*=\ker\mathfrak{b}(T_+^*)=\ker(T_-)$. Now $u\in\ker T_+$ if and only if u satisfies the differential equation

$$0=iu'(s)+if(s)u(s).$$

This is a first-order, one dimensional ODE so all solutions are given by

$$u(s)=Ce^{-F(s)}$$

for $C \in \mathbb{C}$ and F a primitive function for f.

A primitive function for $f(s) = \alpha \tan(\alpha s)$ is $F(s) = -\ln(\cos(\alpha s))$, so the kernel of T_+ is given by constant multiples of $u_+(s) = \cos(\alpha s)$, so $\ker T_+ = \mathbb{C}u_+$. Similarly we find that the kernel of T_- is given by constant multiples of $u_-(s) = \cos(\alpha s)^{-1}$, however u_- is not an $L^2(-\varepsilon, \varepsilon)$ function so $\ker T_- = \{0\}$. Hence $\mathrm{Index}(T) = \dim\ker(T_+) - \dim\ker(T_-) = 1$. $\square$

7 Product of Immersion module with $\mathbb{R}^2$

Now that we have all the ingredients we are ready to prove that $[S^1]\otimes\mathbb{1} = \iota_!\otimes[\mathbb{R}^2]$ at the unbounded level. Let us first give a short overview of the ingredients.

$$\begin{aligned}
\iota_! &= \big[\big(\mathcal{E} = C_0(S^1\times(-\varepsilon,\varepsilon)), S = f(s)\big)\big] \in KK_1(C(S^1), C_0(\mathbb{R}^2)),\\
[S^1] &= \left[\left(L^2(S^1), D_{S^1} = i\frac{\mathrm{d}}{\mathrm{d}\theta}\right)\right] \in KK_1(C(S^1),\mathbb{C}),\\
[\mathbb{R}^2] &= \left[\left(L^2(\mathbb{R}^2,\mathbb{C}^2), D_{\mathbb{R}^2} = i\gamma^1\frac{\partial}{\partial r} + i\frac{1}{r}\gamma^2\frac{\partial}{\partial_\theta} + i\frac{1}{2r}\gamma^1;\gamma^3\right)\right] \in KK_0(C_0(\mathbb{R}^2),\mathbb{C}),\\
\mathbb{1} &= \left[\left(L^2((-\varepsilon,\varepsilon),\mathbb{C}^2), T = i\gamma^1\frac{\mathrm{d}}{\mathrm{d}s} + \gamma^2 f(s);\gamma^3\right)\right] \in KK_0(\mathbb{C},\mathbb{C}).
\end{aligned}$$

We choose our γ-matrices as

$$\gamma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \gamma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \gamma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Before we compute the product $\iota_!\otimes[\mathbb{R}^2]$ we want to make all our cycles even, since that is where the criterion of Kucerovsky [Kuc96] is applicable. To do this we use the doubling procedure described in Lemma 4.4. We get

$$\widetilde{\iota_!} = \left[\left(\tilde{\mathcal{E}} = \mathcal{E}\otimes\mathbb{C}^2, \tilde{S} = S\otimes\gamma^2; 1\otimes\gamma^3\right)\right] \in KK_0(C(S^1)\otimes\mathrm{Cl}_1,\mathbb{C})$$

where $g_1\otimes 1 + g_2\otimes e \in C(S^1)\otimes\mathrm{Cl}_1$ acts via $g_1\otimes 1 + g_2\otimes\gamma^1$. The connection $\nabla^{\mathcal{E}}$ is doubled to $\widetilde{\nabla^{\mathcal{E}}} = \nabla^{\mathcal{E}}\otimes 1$. Similarly we need to make $[S^1]$ even by the same process, yielding

$$\widetilde{[S^1]} = \left[\left(L^2(S^1)\otimes\mathbb{C}^2, i\frac{\mathrm{d}}{\mathrm{d}\theta}\otimes\gamma^2; 1\otimes\gamma^3\right)\right] \in KK_0(C(S^1)\otimes\mathrm{Cl}_1,\mathbb{C}).$$

We will now first, in Section 7.1, compute the product of $\tilde{\iota_!}$ and $[\mathbb{R}^2]$ using the connection we have on $\iota_!$. Since our cycles do not satisfy the assumptions in [KL13], we cannot immediately claim that the product operator we find will be self-adjoint or compactly resolved. Therefore we will prove that the product does have these properties in Sections 7.2 and 7.3 respectively.

Then in Section 7.4 we use Kucerovsky's conditions to verify that our product based on the connection indeed represent the product of $\widetilde{\iota_!}$ and $[\mathbb{R}^2]$. Finally we show that our product cycle also represents the product of $\widetilde{[S^1]}$ and $\mathbb{1}$ in Section 7.5.

7.1 Form of the product operator

In order to compute the product of $\widetilde{\iota_!}$ and $[\mathbb{R}^2]$ we first simplify the balanced tensor product $\tilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^2)} L^2(\mathbb{R}^2, \mathbb{C}^2)$ into a single Hilbert space, so that we can express our product operators as operators on this simplified Hilbert space.

Lemma 7.1. *The balanced tensor product of $\tilde{\mathcal{E}}$ and $L^2(\mathbb{R}^2, \mathbb{C}^2)$ is unitarily equivalent as graded Hilbert $C(S^1)$-$\mathbb{C}$ bimodule to $L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with grading $1 \otimes \gamma^3 \otimes \gamma^3$.*

Proof. This follows directly from unitarity of the map

$$\begin{aligned} U' &: \mathcal{E} \otimes_{C_0(\mathbb{R}^2)} L^2(\mathbb{R}^2) \to L^2(S^1 \times (-\varepsilon, \varepsilon)), \\ U'&(g \otimes \psi)(\theta, s) = g(\theta, s)\psi(\theta, s+1), \end{aligned}$$

by $L^2(\mathbb{R}^2, \mathbb{C}^2) = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and setting

$$U(g \otimes v \otimes \psi \otimes w) = U'(g \otimes \psi) \otimes v \otimes w.$$

Let us check that U' is indeed a unitary.

$$\begin{aligned} \langle U'(g \otimes \psi), &U'(g' \otimes \psi')\rangle_{L^2(S^1 \times (-\varepsilon, \varepsilon))} \\ &= \int_{S^1 \times (-\varepsilon,\varepsilon)} \overline{g(\theta, s)\psi(\theta, s+1)} g'(\theta, s)\psi'(\theta, s+1) \mathrm{d}s\mathrm{d}\theta, \\ &= \int_{S^1 \times (1-\varepsilon, 1+\varepsilon)} \overline{\psi(\theta, r)} \frac{1}{r} \overline{g(\theta, r-1)} g'(\theta, r-1)\psi'(\theta, r) r\mathrm{d}r\mathrm{d}\theta, \\ &= \int_{\mathbb{R}^2} \overline{\psi(\theta, r)}\, \langle g, g'\rangle_{\mathcal{E}}(\theta, r)\, \psi'(\theta, r) r\mathrm{d}r\mathrm{d}\theta, \\ &= \langle \psi, \langle g, g'\rangle_{\mathcal{E}} \cdot \psi'\rangle_{L^2(\mathbb{R}^2)}, \\ &= \langle g \otimes \psi, g' \otimes \psi'\rangle_{\mathcal{E} \otimes L^2(\mathbb{R}^2)}. \end{aligned}$$

Furthermore U' is surjective since $\chi_{S^1 \times (1-\varepsilon, 1+\varepsilon)} \in L^2(\mathbb{R}^2)$ so that the L^2-dense set $C_0(S^1 \times (-\varepsilon, \varepsilon))$ is in the range of U'. □

Now that we have a convenient space for the product operator to act on it is time to compute the product operator itself, using Definition 4.12 and the connection $\nabla^{\mathcal{E}}$.

Proposition 7.2. *Write U for the unitary equivalence from Lemma 7.1. The product operator is*

$$U(S \otimes 1 + \gamma^3 \otimes_{\nabla^{\mathcal{E}}} D_{\mathbb{R}^2})U^* = f(s) \otimes \gamma^2 \otimes 1 + i\partial_s \otimes \gamma^3 \otimes \gamma^1 + i\frac{1}{1+s}\partial_\theta \otimes \gamma^3 \otimes \gamma^2.$$

Proof. We will start with $U(\tilde{S}\otimes 1)U^*$. It will be convenient to write $L^2(\mathbb{R}^2,\mathbb{C}^2)\cong L^2(\mathbb{R}^2)\otimes\mathbb{C}^2$. Let $g\otimes v\otimes\psi\otimes w\in\mathcal{E}\otimes\mathbb{C}^2\otimes L^2(\mathbb{R}^2)\otimes\mathbb{C}^2$. Then

$$\tilde{S}\otimes 1 = S\otimes\gamma^2\otimes 1_{L^2(\mathbb{R}^2)}\otimes 1_{\mathbb{C}^2}$$

so

$$\begin{aligned}
U(\tilde{S}\otimes 1)(g\otimes v\otimes\psi\otimes w)(\theta,s) &= U(Sg\otimes\gamma^2 v\otimes\psi\otimes w)(\theta,s),\\
&= f(s)g(\theta,s)\psi(\theta,s+1)\otimes\gamma^2 v\otimes w,\\
&= \big(f(s)\otimes\gamma^2\otimes 1_{\mathbb{C}^2}\big)\,U(g\otimes v\otimes\psi\otimes w)(\theta,s).
\end{aligned}$$

Hence $U(\tilde{S}\otimes 1)U^* = f(s)\otimes\gamma^2\otimes 1_{\mathbb{C}^2}$.

Using a similar approach we will compute $U(\gamma^3\otimes_{\nabla^{\mathcal{E}}} D_{\mathbb{R}^2})U^*$.

$$\begin{aligned}
\gamma^3\otimes_{\nabla^{\mathcal{E}}} D_{\mathbb{R}^2}(g\otimes v\otimes\psi\otimes w) =& g\otimes\gamma^3 v\otimes D_{\mathbb{R}^2}(\psi\otimes w)+\nabla^{\mathcal{E}}(g\otimes\gamma^3 v)\cdot(f\otimes w),\\
=& g\otimes\gamma^3 v\otimes\left(i\partial_r\psi\otimes\gamma^1 w+i\frac{1}{r}\partial_\theta\psi\otimes\gamma^2 w+i\frac{1}{2r}\psi\otimes\gamma^1 w\right)\\
&+\nabla^{\mathcal{E}}_{\partial_r}(g)\otimes\gamma^3 v\otimes(ic(\mathrm{d}r)(\psi\otimes w))\\
&+\nabla^{\mathcal{E}}_{\partial_\theta}(g)\otimes\gamma^3 v\otimes(ic(\mathrm{d}\theta)(\psi\otimes w)),\\
=& ig\otimes\gamma^3 v\otimes\left(\partial_r\psi\otimes\gamma^1 w+\frac{1}{r}\partial_\theta\psi\otimes\gamma^2 w+\frac{1}{2r}\psi\otimes\gamma^1 w\right)\\
&+\left(\partial_s g-\frac{1}{2(s+1)}g\right)\otimes\gamma^3 v\otimes\psi\otimes i\gamma^1 w\\
&+(\partial_\theta g)\otimes\gamma^3 v\otimes\frac{1}{r}\psi\otimes i\gamma^2 w.
\end{aligned}$$

For notational purposes we now move both the $\mathbb{C}^2$ components to the right, then

$$\begin{aligned}
\gamma^3\otimes_{\nabla^{\mathcal{E}}} D_{\mathbb{R}^2}(g\otimes\psi\otimes v\otimes w) =& i\,(g\otimes\partial_r\psi+\partial_s g\otimes\psi)\otimes\gamma^3 v\otimes\gamma^1 w\\
&+i\left(g\otimes\frac{1}{2r}\psi-\frac{1}{2(s+1)}g\otimes\psi\right)\otimes\gamma^3 v\otimes\gamma^1 w\\
&+i\left(g\otimes\frac{1}{r}\partial_\theta\psi+\partial_\theta g\otimes\frac{1}{r}\psi\right)\otimes\gamma^3 v\otimes\gamma^2 w.
\end{aligned}$$

The second term in this expression is 0, since we may move the $\frac{1}{2r}$ over the tensorproduct at the cost of changing r into $s+1$, since that is how $C_0(\mathbb{R}^2)$ acts on $\mathcal{E}$. Under the map U the remaining terms equal

$$\begin{aligned}
U(\gamma^3\otimes_{\nabla^{\mathcal{E}}} D_{\mathbb{R}^2})(g\otimes\psi\otimes v\otimes w) =& i\partial_s U'(g\otimes\psi)\otimes\gamma^3 v\otimes\gamma^1 w\\
&+i\frac{1}{1+s}\partial_\theta U'(g\otimes\psi)\otimes\gamma^3 v\otimes\gamma^2 w,
\end{aligned}$$

where we use the U' from Lemma 7.1. Therefore

$$U(\gamma^3 \otimes_{\nabla^\varepsilon} D_{\mathbb{R}^2})U^* = i\partial_s \otimes \gamma^3 \otimes \gamma^1 + i\frac{1}{1+s}\partial_\theta \otimes \gamma^3 \otimes \gamma^2.$$

Finally, we arrive at

$$U(S \otimes 1 + \gamma^3 \otimes_{\nabla^\varepsilon} D_{\mathbb{R}^2})U^* = f(s) \otimes \gamma^2 \otimes 1_{\mathbb{C}^2} + i\partial_s \otimes \gamma^3 \otimes \gamma^1 + i\frac{1}{1+s}\partial_\theta \otimes \gamma^3 \otimes \gamma^2$$

□

Our eventual goal is to show that this operator represents the product of $\widetilde{[S^1]}$ and $\mathbb{1}$. The operator for the product of $[\tilde{S^1}]$ and $\mathbb{1}$ should be of the form

$$\tilde{D}_{S^1} \otimes 1 + \gamma^3 \otimes T,$$

there is no connection since this tensor product is only balanced over $\mathbb{C}$. Since $\tilde{D}_{S^1} = D_{S^1} \otimes \gamma^2$ and writing $T = i\partial_s\gamma^1 + f(s)\gamma^2$ we see that we are close but not quite there yet. We would like the ∂_θ term to have a $\gamma^2 \otimes 1$ while we would like the $f(s)$ term to have a $\gamma^3 \otimes \gamma^2$. Fortunately, this is possible.

Lemma 7.3. *There is a unitary transformation $V : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ such that*

$$V(\gamma^3 \otimes \gamma^3)V^* = \gamma^3 \otimes \gamma^3, \qquad V(\gamma^2 \otimes 1)V^* = \gamma^3 \otimes \gamma^2,$$
$$V(\gamma^3 \otimes \gamma^1)V^* = \gamma^3 \otimes \gamma^1, \qquad V(\gamma^3 \otimes \gamma^2)V^* = \gamma^2 \otimes 1.$$

Proof. A set of straightforward calculations shows that the map $V = \frac{1}{\sqrt{2}}(\gamma^3 \otimes 1 + \gamma^2 \otimes \gamma^2)$ works. □

Notation 7.4. *We write*

$$\begin{aligned} D_\times &= VU(\tilde{S} \times 1 + \gamma^3 \otimes_{\tilde{\nabla}^\varepsilon} D_{\mathbb{R}^2})U^*V^*, \\ &= f(s) \otimes \gamma^3 \otimes \gamma^2 + i\partial_s \otimes \gamma^3 \otimes \gamma^1 + i\frac{1}{1+s}\partial_\theta \otimes \gamma^2 \otimes 1. \end{aligned}$$

and call $D_\times$ the product operator.

Proposition 7.5. *The triple*

$$\big(L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, D_\times; 1 \otimes \gamma^3 \otimes \gamma^3\big)$$

defines an unbounded $KK_0(C(S^1), \mathbb{C})$ cycle. (The domain of $D_\times$ will be defined in Corollary 7.12.)

Proof. Since $L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \tilde{\mathcal{E}} \otimes_{C_0(\mathbb{R}^2)} L^2(\mathbb{R}^2, \mathbb{C}^2)$ we know from Theorem 2.16 that $L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is a graded Hilbert $C(S^1) \otimes \mathbb{C}\mathrm{l}_1$-$\mathbb{C}$ bimodule. A simple check shows that $D_\times$ is odd, so we have to show that $D_\times$ is self adjoint, which we do in Corollary 7.12, and has compact resolvent which we do in Proposition 7.17. □

Note that along the circle, where $s = 0$, we have that $D_\times$ is the sum of the Dirac operator on S^1 with an operator representing the $KK_0(\mathbb{C}, \mathbb{C})$ multiplicative unit, as announced in the introduction.

Remark 7.6. *Unfortunately, our cycles do not satisfy the assumptions of Theorem 4.13 since*

$$\begin{aligned}[][S \otimes 1, \gamma \otimes_\nabla D_{\mathbb{R}^2}](i + S \otimes 1)^{-1} &= \frac{\partial f(s)}{\partial s} \cdot \frac{1}{i + f(s)}, \\ &= \frac{\alpha^2}{\cos(\alpha s)} \cdot \frac{1}{i \cos(\alpha s) + \alpha \sin(\alpha s)}\end{aligned}$$

is unbounded. That this is not a consequence of our choice for f can be seen as follows.

Suppose $f' \leq Cf$ for some C, which is equivalent to relative boundedness of the commutator, then $f' = g \cdot f$ with $|g| \leq C$. The solution to this differential equation is $f = e^G$ with G a primitive for g, but $G(x) \leq G(0) + Cx$ so $f(x) \leq e^{G(0)+Cx}$ which means f cannot go to infinity in finite time as we require. For further discussion see Section 8.

7.2 Self-adjointness of product operator

For this section and the next we want to rearrange our Hilbert space $L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ once more. We will use

$$L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong L^2((-\varepsilon, \varepsilon), \mathbb{C}^2) \otimes L^2(S^1, \mathbb{C}^2)$$

such that under this identification

$$\begin{aligned} D_\times &= \left(f(s) \otimes \gamma^2 + i\partial_s \otimes \gamma^1\right) \otimes \gamma^3 + \frac{1}{1+s} \otimes \left(i\gamma^2 \partial_\theta\right), \\ &= \begin{pmatrix} 0 & i\partial_s - if(s) \\ i\partial_s + if(s) & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{1+s} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \partial_\theta \\ -\partial_\theta & 0 \end{pmatrix}. \end{aligned}$$

This first term is the operator T from Section 6, which we know to be essentially self-adjoint on the domain $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ by Proposition 6.2. Also $\left(\begin{smallmatrix} 0 & \partial_\theta \\ -\partial_\theta & 0 \end{smallmatrix}\right)$ is essentially self-adjoint on $C^\infty(S^1, \mathbb{C}^2)$. The other factors are bounded, self adjoint operators which leads us to investigate operators of the form $D_1 \otimes A + B \otimes D_2$ where D_1, D_2, A, B are all self-adjoint and A, B are bounded.

Intuitively the idea is that the bounded operators do not influence the domains, similar to how $D_1 + B$ would be self-adjoint on $\mathrm{dom}(D_1)$ and that D_1 and D_2 do not influence each other in a relevant way since they act on separate tensor product factors. However, to make this precise we will use a concept from van den Dungen [Dun16].

Definition 7.7. *Let $D : \mathrm{dom}(D) \to H$ be a densely defined symmetric operator on some Hilbert space H. An* adequate approximate identity *for D is a sequential approximate identity $\{\phi_k\}_{k\in\mathbb{N}}$ on H such that $\phi_k \,\mathrm{dom}(D^*) \subset \mathrm{dom}(\overline{D})$, $[\overline{D}, \phi_k]$ is bounded on $\mathrm{dom}(D)$ and $\sup_{k\in\mathbb{N}} ||\overline{[\overline{D}, \phi_k]}|| < \infty$.*

Remark 7.8. *Van den Dungen gives the previous definition in the context of Hilbert modules. We will keep ourselves to the Hilbert space case, even though Proposition 7.9 continues to hold in the Hilbert module setting, albeit with a more intricate proof.*

The following result is the reason we introduce these adequate approximate identities.

Proposition 7.9. *Let $D : \mathrm{dom}(D) \to H$ be a densely defined symmetric operator on a Hilbert space H, and suppose $\{\phi_k\}_{k\in\mathbb{N}}$ is an adequate approximate identity for D. Then D is essentially self-adjoint.*

Proof. See [Dun16]. □

We also have the, much easier, converse.

Lemma 7.10. *Suppose $D : \mathrm{dom}(D) \to H$ is a self-adjoint operator. Then $\phi_k := D(1 + \frac{1}{k^2}D^2)^{-1}$ defines an adequate approximate identity $\{\phi_k\}_{k\in\mathbb{N}}$ for D. Furthermore $||(1 + \frac{1}{k^2}D^2)^{-1}|| \leq 1$ and $||D(1 + \frac{1}{k^2}D^2)^{-1}|| \leq k$.*

Proof. The norm-estimates, as well as the fact that $\{\phi_k\}$ defines an approximate unit, are in [Ped89, Thm 5.1.9]. Furthermore, this theorem tells us that $[D, \phi_k] = 0$ on $\mathrm{dom}(D)$. The only remaining requirement is then that $\phi_k \,\mathrm{dom}(D) \subset \mathrm{dom}(D)$, we even have the stronger result that $\phi_k H \subset \mathrm{dom}(D)$ since $\phi_k = k^2(D+ki)^{-1}(D-ki)^{-1}$ and the resolvents map H into $\mathrm{dom}(D)$. □

We are now ready to return to our case of interest.

Proposition 7.11. *Let $D_1 : \mathrm{dom}(D_1) \to H_1$ and $D_2 : \mathrm{dom}(D_2) \to H_2$ be densely defined self-adjoint operators on Hilbert spaces H_1 and H_2. Let $A : H_2 \to H_2$ and*

$B: H_1 \to H_1$ be bounded, self-adjoint operators, such that $||(1+\frac{1}{k^2}D_1^2)^{-1}[B, D_1^2](1+\frac{1}{k^2}D_1^2)^{-1}|| \leq c_1 k$ and $||(1+\frac{1}{k^2}D_2^2)^{-1}[A, D_2^2](1+\frac{1}{k^2}D_2^2)^{-1}|| \leq c_2 k$ for some $c_1, c_2 \in \mathbb{R}$. Then $D_1 \otimes A + B \otimes D_2$ is essentially self-adjoint on $\mathrm{dom}(D_1) \otimes_{alg} \mathrm{dom}(D_2)$.

Proof. We will show that

$$\phi_k := (1 + \frac{1}{k^2}D_1^2)^{-1} \otimes (1 + \frac{1}{k^2}D_2^2)^{-1}$$

is an adequate approximate identity for $D_1 \otimes A + B \otimes D_2$, and then invoke Proposition 7.9. For ease of notation introduce $a = D_1 \otimes A + B \otimes D_2$.

Let us first show that ϕ_k is an approximate identity for $H_1 \otimes H_2$. Since $\{(1 + \frac{1}{k^2}D_1^2)^{-1})\}_{k\in\mathbb{N}}$ and $\{(1 + \frac{1}{k^2}D_2^2)^{-1})\}_{k\in\mathbb{N}}$ are approximate identities for H_1 and H_2 by Lemma 7.10, ϕ_k is clearly an approximate unit for elementary tensors. Suppose $z = \lim_n z_n$, $z \in H_1 \otimes H_2$ and $z_n \in H_1 \otimes_{alg} H_2$. Then

$$\begin{aligned} ||z - \phi_k z|| &= ||z - z_n + z_n - \phi_k z_n + \phi_k z_n - \phi_k z||, \\ &\leq ||z - z_n|| + ||z_n - \phi_k z_n|| + ||\phi_k|| \cdot ||z_n - z||. \end{aligned}$$

For any $\epsilon > 0$ we can find an N such that $||z - z_N|| < \frac{\epsilon}{3}$, and for z_N we can find a K such that $||z_N - \phi_k z_N|| < \frac{\epsilon}{3}$ for all $k \geq K$. Since $||\phi_k|| = ||(1 + \frac{1}{k^2}D_1^2)^{-1}|| \cdot ||(1 + \frac{1}{k^2}D_2^2)^{-1}|| \leq 1$, we get for $k \geq K$ that $||z - \phi_k z|| < \epsilon$ so that ϕ_k is an approximate unit.

Now we show that $\phi_k \,\mathrm{dom}(a^*) \subset \mathrm{dom}(\overline{a})$, in fact we will show the stronger $\phi_k H_1 \otimes H_2 \subset \mathrm{dom}(\overline{a})$ similar to what we saw in Lemma 7.10. We will use that $a\phi_k$ is bounded, indeed

$$\begin{aligned} ||a\phi_k|| &= ||(D_1 \otimes A + B \otimes D_2)(1 + \frac{1}{k^2}D_1^2)^{-1} \otimes (1 + \frac{1}{k^2}D_2^2)^{-1}||, \\ &= ||D_1(1 + \frac{1}{k^2}D_1^2)^{-1} \otimes A(1 + \frac{1}{k^2}D_2^2)^{-1} + B(1 + \frac{1}{k^2}D_1^2)^{-1} \otimes D_2(1 + \frac{1}{k^2}D_2^2)^{-1}||, \\ &\leq ||D_1(1 + \frac{1}{k^2}D_1^2)^{-1} \otimes A(1 + \frac{1}{k^2}D_2^2)^{-1}|| + ||B(1 + \frac{1}{k^2}D_1^2)^{-1} \otimes D_2(1 + \frac{1}{k^2}D_2^2)^{-1}||, \\ &= ||D_1(1 + \frac{1}{k^2}D_1^2)^{-1}|| \cdot ||A(1 + \frac{1}{k^2}D_2^2)|| + ||B(1 + \frac{1}{k^2}D_1^2)^{-1}|| \cdot ||D_2(1 + \frac{1}{k^2}D_2^2)^{-1}||, \\ &\leq k||A|| + ||B||k. \end{aligned}$$

Let again $z \in H_1 \otimes H_2$, $z = \lim_n z_n$, $z_n \in H_1 \otimes_{alg} H_2$ and fix k. Clearly $\phi_k z_n \in \mathrm{dom}(a) = \mathrm{dom}(D_1) \otimes_{alg} \mathrm{dom}(D_2)$ and $\phi_k z_n \to \phi_k z$ since ϕ_k is bounded. Moreover, as we just saw, $a\phi_k$ is bounded so that $a\phi_k z_n \to a\phi_k z$. Hence $\phi_k z \in \mathrm{dom}(\overline{a})$, since we have a sequence in $\mathrm{dom}(a)$ for which the images under a also converge.

Finally we consider $[\overline{a}, \phi_k]$ on $\mathrm{dom}(a)$, and show that these commutators are bounded uniformly in k. Recall from the proof of Lemma 7.10 that D_i and $(1+\frac{1}{k^2}D_i^2)^{-1}$ commute on $\mathrm{dom}(D_i)$. Then

$$\begin{aligned}
[\overline{a}, \phi_k] =& [D_1 \otimes A + B \otimes D_2, (1+\frac{1}{k^2}D_1^2)^{-1} \otimes (1+\frac{1}{k^2}D_2^2)^{-1}],\\
=& [D_1 \otimes A, (1+\frac{1}{k^2}D_1^2)^{-1} \otimes (1+\frac{1}{k^2}D_2^2)]\\
&\quad + [B \otimes D_2, (1+\frac{1}{k^2}D_1^2)^{-1} \otimes (1+\frac{1}{k^2}D_2^2)^{-1}],\\
=& [D_1, (1+\frac{1}{k^2}D_1^2)^{-1}] \otimes A(1+\frac{1}{k^2}D_2^2)^{-1}\\
&\quad + D_1(1+\frac{1}{k^2}D_1^2)^{-1} \otimes [A, (1+\frac{1}{k^2}D_2^2)^{-1}]\\
&\quad + [B, (1+\frac{1}{k^2}D_1^2)^{-1}] \otimes D_2(1+\frac{1}{k^2}D_2^2)^{-1}\\
&\quad + B(1+\frac{1}{k^2}D_1^2)^{-1} \otimes [D_2, (1+\frac{1}{k^2}D_2^2)^{-1}],\\
=& D_1(1+\frac{1}{k^2}D_1^2)^{-1} \otimes [A, (1+\frac{1}{k^2}D_2^2)^{-1}]\\
&\quad + [B, (1+\frac{1}{k^2}D_1^2)^{-1}] \otimes D_2(1+\frac{1}{k^2}D_2^2)^{-1}.
\end{aligned}$$

Since $||D_i(1+\frac{1}{k^2}D_i^2)^{-1}|| \leq k$ we want to bound $[A, (1+\frac{1}{k^2}D_2^2)^{-1}]$ and $[B, (1+\frac{1}{k^2}D_1^2)^{-1}]$ by something of order $\frac{1}{k}$.

First note that

$$\begin{aligned}
[B, (1+\frac{1}{k^2}D_1^2)^{-1}] &= (1+\frac{1}{k^2}D_1^2)^{-1}[1+\frac{1}{k^2}D_1^2, B](1+\frac{1}{k^2}D_1^2)^{-1},\\
&= -\frac{1}{k^2}(1+\frac{1}{k^2}D_1^2)^{-1}[B, D_1^2](1+\frac{1}{k^2}D_1^2)^{-1}.
\end{aligned}$$

By assumption there exists a c such that

$$||(1+\frac{1}{k^2}D_1^2)^{-1}[B, D_1^2](1+\frac{1}{k^2}D_1^2)^{-1}|| \leq c_1 k$$

which implies

$$||[B, (1+\frac{1}{k^2}D_1^2)^{-1}]|| \leq c_1 \frac{1}{k}.$$

By the same reasoning we get

$$||[A, (1+\frac{1}{k^2}D_2^2)^{-1}]|| \leq c_2 \frac{1}{k}.$$

Together this implies $||[\overline{a}, \phi_k]|| \leq c_1 + c_2$.

Now that we have established that ϕ_k is an adequate approximate identity for a, we find that a is essentially self-adjoint by Proposition 7.9. $\square$

Corollary 7.12. *The operator* $D_\times$ *is essentially self-adjoint on domain* $\mathrm{dom}(D_\times) = \mathrm{dom}(T) \otimes_{alg} \mathrm{dom}(i\gamma^2\partial_\theta)$.

Proof. Referring to the notation of Proposition 7.11 we have $D_1 = T$, $A = \gamma^3$, $B = \frac{1}{1+s} 1_{\mathbb{C}^2}$ and $D_2 = i\gamma^2\partial_\theta$. Let us compute the relevant commutators.

$$\begin{aligned} [A, D_2^2] &= [\gamma^3, (i\gamma^2\partial_\theta)^2], \\ &= [\gamma^3, -\partial_\theta^2], \\ &= 0. \end{aligned}$$

$$\begin{aligned} [B, D_1^2] &= \left[\frac{1}{1+s} 1_{\mathbb{C}^2}, \begin{pmatrix} -\partial_s^2 + f(s)^2 - f'(s) & 0 \\ 0 & -\partial_s^2 + f(s)^2 + f'(s) \end{pmatrix} \right], \\ &= \left[\frac{1}{1+s}, -\partial_s^2 \right] 1_{\mathbb{C}^2}, \\ &= \left(\frac{2}{(1+s)^3} + \frac{2}{(1+s)^2} \partial_s \right) 1_{\mathbb{C}^2}. \end{aligned}$$

We will prove that $\partial_s(1 + \frac{1}{k^2}T^2)^{-1}$ is bounded by ck for some c. From Lemma 6.9 we know that for $\lambda = \pm\alpha$ we have $||(T + \lambda i)\psi|| \geq ||\psi'||$. This also holds for $|\lambda| \geq \alpha$ since T is symmetric. In particular $||(T + mi)\psi|| \geq ||\psi'||$ for all $m \in \mathbb{Z} \setminus \{0\}$. Furthermore $||(T + mi)\psi|| \geq m||\psi||$ for all $m \in \mathbb{Z}$ and $\psi \in \mathrm{dom}(T)$, again by symmetry of T.

Let ϕ be arbitrary in $L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$ and $\phi = (1 + \frac{1}{k^2}T^2)\psi$, this is possible since $(1 + \frac{1}{k^2}T^2) = \frac{1}{k^2}(T - ki)(T + ki)$ and these are surjective. Consider

$$\begin{aligned} ||\phi|| &= ||(1 + \frac{1}{k^2}T^2)\psi||, \\ &= \frac{1}{k^2}||(T - ki)(T + ki)\psi||, \\ &\geq \frac{1}{k^2}k||(T + ki)\psi||, \\ &\geq \frac{1}{k}||\psi'||. \end{aligned}$$

Then

$$\begin{aligned}||\partial_s(1+\frac{1}{k^2}T^2)^{-1}\phi|| &= ||\partial_s\psi||,\\ &= ||\psi'||,\\ &\leq k||\phi||.\end{aligned}$$

So $||\partial_s(1+\frac{1}{k^2}T^2)^{-1}|| \leq k$, which in turn means that

$$\begin{aligned}||(1+\frac{1}{k^2}D_1^2)^{-1}[B,D_1^2](1+\frac{1}{k^2}D_1^2)^{-1}|| &\leq 1\cdot\left(\frac{2}{(1-\varepsilon)^3}\cdot 1+\frac{2}{(1-\varepsilon)^2}\cdot k\right),\\ &\leq \frac{4}{(1-\varepsilon)^3}k.\end{aligned}$$

Therefore Proposition 7.11 applies. □

Corollary 7.13. *If D_1 and D_2 are essentially self-adjoint on* $\mathrm{dom}(D_1)$ *and* $\mathrm{dom}(D_2)$ *and satisfy the assumptions in Proposition 7.11, then $D_1\otimes A+B\otimes D_2$ is essentially self-adjoint on* $\mathrm{dom}(D_1)\otimes_{alg}\mathrm{dom}(D_2)$.

Proof. Write $\mathrm{dom}(\overline{D_1})$ and $\mathrm{dom}(\overline{D_2})$ for the domains of self-adjointness of D_1 and D_2. Then we know that $D_1\otimes A+B\otimes D_2$ is essentially self-adjoint on $\mathrm{dom}(\overline{D_1})\otimes_{alg}\mathrm{dom}(\overline{D_2})$. Write a_0 for the closure of $D_1\otimes A+B\otimes D_2$ defined on $\mathrm{dom}(D_1)\otimes_{alg}\mathrm{dom}(D_2)$ and a for the closure on $\mathrm{dom}(\overline{D_1})\otimes_{alg}\mathrm{dom}(\overline{D_2})$.

Clearly $a_0\subset a$, so we want to show that $a\subset a_0$. This follows if we can show that $\mathrm{dom}(\overline{D_1})\otimes_{alg}\mathrm{dom}(\overline{D_2})\subset\overline{\mathrm{dom}(D_1)\otimes_{alg}\mathrm{dom}(D_2)}$, with the closure taken in the graph-norm of a. So suppose $\psi\otimes\phi\in\mathrm{dom}(\overline{D_1})\otimes_{alg}\mathrm{dom}(\overline{D_2})$. Then $\psi=\lim x_n$, $\phi=\lim y_n$ such that $D_1\psi=\lim D_1x_n$ and $D_2\phi=\lim D_2y_n$, with $x_n\in\mathrm{dom}(D_1)$ and $y_n\in\mathrm{dom}(D_2)$ since the D_i are essentially self-adjoint on the $\mathrm{dom}(D_i)$. But then

$$\begin{aligned}||a(x_n\otimes y_n)-&a(\psi\otimes\phi)||\\ &=||(D_1\otimes A)(x_n\otimes y_n-\psi\otimes\phi)+(B\otimes D_2)(x_n\otimes y_n-\psi\otimes\phi)||,\\ &\leq||D_1x_n\otimes Ay_n-D_1\psi\otimes Ay_n||+||D_1\psi\otimes Ay_n-D_1\psi\otimes A\phi||\\ &\quad+||Bx_n\otimes D_2y_n-Bx_n\otimes D_2\phi||+||Bx_n\otimes D_2\phi-B\psi\otimes D_2\phi||,\\ &\leq||D_1(x_n-\psi)||\cdot||Ay_n||+||D_1\psi||\cdot||A(y_n-\phi)||\\ &\quad+||Bx_n||\cdot||D_2(y_n-\phi)||+||B(x_n-\psi)||\cdot||D_2\phi||\end{aligned}$$

tends to zero. Therefore $\psi\otimes\phi\in\overline{\mathrm{dom}(D_1)\otimes_{alg}\mathrm{dom}(D_2)}$ (closure in the graph norm) so that $a\subset a_0$. □

A remark about self-adjointness of $\gamma \otimes_\nabla D_{\mathbb{R}^2}$ is in order here. In [KL13] self-adjointness of the product operator is proven by showing that $D_1 \otimes 1$ and $\gamma \otimes_\nabla D_2$ separately are (essentially) self-adjoint and that they anti-commute, which then proves that their sum is again (essentially) self-adjoint.

In our case $\gamma \otimes_\nabla D_{\mathbb{R}^2}$ is not essentially self-adjoint on the domain $C_0^\infty(S^1 \times (-\varepsilon, \varepsilon))$ (with the appropriate unitary transformations and added spinor components). It is however essentially self-adjoint on domains of the form $\mathrm{dom}_\phi = \{f \in C^\infty(S^1 \times (-\varepsilon, \varepsilon)) | f(-\varepsilon) = e^{i\phi} f(\varepsilon)\}$ for some $\phi \in \mathbb{R}$. This can be proven using Proposition 7.11 and [Lax02, Example 33.3]. These domains dom_ϕ are not the domain specified in [KL13], which is $C_0^\infty(S^1 \times (-\varepsilon, \varepsilon))$, neither is there any obvious choice of ϕ.

7.3 Compact Resolvent of product operator

To prove that the resolvent of $D_\times$ is compact we invoke some fairly heavy machinery, in the form of the min-max principle.

Theorem 7.14 (min-max principle)**.** *Let $D : \mathrm{dom}(D) \to H$ be a self-adjoint operator that is bounded below. Then D has compact resolvent if and only if $\mu_n(D) \to \infty$ as $n \to \infty$, where*

$$\mu_n(D) = \sup_{\phi_1, \ldots, \phi_{n-1}} U_D(\phi_1, \ldots, \phi_{n-1}),$$
$$U_D(\phi_1, \ldots, \phi_m) = \inf_{\psi \in \mathrm{dom}(D), ||\psi||=1, \psi \perp \phi_k \forall k} \langle \psi, A\psi \rangle.$$

Proof. See theorems XIII.1 and XIII.64 in [RS80]. ☐

We will also use the following characterization of compact resolvents.

Theorem 7.15. *Let $D : \mathrm{dom}(D) \to H$ be a self-adjoint operator that is bounded below. Then D has compact resolvent if and only if there exists a complete orthonormal basis of eigenvectors $\{\phi_n\}_{n \in \mathbb{N}}$, $\phi_n \in \mathrm{dom}(D)$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_n \to \infty$*

Proof. See theorem XIII.64 in [RS80]. ☐

We will apply the min-max principle to $D_\times^2$, which is positive, and hence bounded below, since it is the square of a self-adjoint operator. However, before we do we want to prove a link between the eigenvectors of the square of a compactly resolved self-adjoint operator and the operator itself.

Proposition 7.16. *Let $D : \mathrm{dom}(D) \to H$ be a self-adjoint operator such that $D^2 : \mathrm{dom}(D^2) \to H$ has compact resolvent. Then D also has compact resolvent.*

Furthermore there exists a complete orthonormal basis of eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ *such that* $D\phi_n = \nu_n\phi_n$ *and* $D^2\phi_n = \nu_n^2\phi_n$*, with* $\nu_n^2 \to \infty$.

Proof. The operator D^2 is self-adjoint and bounded below, being the square of a self-adjoint operator, and has compact resolvent by assumption. Therefore we get a complete orthonormal basis of eigenvectors $\{\tilde{\phi}_n\}$ by Theorem 7.15.

Let E_λ be the eigenspace corresponding to an eigenvalue λ, then $\dim(E_\lambda) < \infty$ since $\lambda_n \to \infty$. Since D and D^2 commute, D preserves the eigenspaces E_λ. Moreover, $E_\lambda \subset \mathrm{dom}(D^2)$ so $D|_{E_\lambda}$ is a self-adjoint operator on a finite dimensional space, hence diagonalizable.

Let $\{\phi_n\}$ be the transformation of $\{\tilde{\phi}_n\}$ such that $D|_{E_\lambda}$ is diagonal for all eigenvalues λ. Then $D\phi_n = \nu_n\phi_n$ and clearly $\nu_n^2 = \lambda_n$. $\square$

Proposition 7.17. *The operator* $D_\times^2$ *has compact resolvent.*

Proof. We will show that $\mu_n(D_\times) \to \infty$ and invoke Theorem 7.14.

Recall that $D_\times = T \otimes \gamma^3 + \frac{1}{1+s} \otimes i\gamma^2\partial_\theta$, so that

$$\begin{aligned} D_\times^2 &= T^2 \otimes 1_{\mathbb{C}^2} + \frac{1}{(1+s)^2} \otimes \left(-\partial_\theta^2\right) + \left[T, \frac{1}{1+s}\right] \otimes \gamma^1\partial_\theta, \\ &= T^2 \otimes 1_{\mathbb{C}^2} + \frac{1}{(1+s)^2} \otimes \left(-\partial_\theta^2\right) + -i\gamma^1 \frac{1}{(1+s)^2} \otimes \gamma^1\partial_\theta. \end{aligned}$$

Since T has compact resolvent, so does T^2 which means that by Theorem 7.15 there is a complete orthonormal basis of eigenvectors $\{\psi_n\}_{n\in\mathbb{N}} \subset L^2((-\varepsilon,\varepsilon),\mathbb{C}^2)$ for T with eigenvalues λ_n. Similarly we get a complete orthonormal basis of eigenvectors $\{\phi_n\}_{n\in\mathbb{N}} \subset L^2(S^1,\mathbb{C}^2)$ for $i\gamma^2\partial_\theta$ with eigenvalues ν_n. We also have that the sequences λ_n^2 and ν_n^2 are increasing and tend to infinity.

The set $\{\psi_k \otimes \phi_l\}_{(k,l)\in\mathbb{N}\times\mathbb{N}}$ is a complete orthonormal set for $L^2((-\varepsilon,\varepsilon),\mathbb{C}^2) \otimes L^2(S^1,\mathbb{C}^2)$, using this set we will show that $\mu_n(D_\times^2) \to \infty$. It is clear from the definition of $\mu_n(D_\times^2)$ that the $\mu_n(D_\times^2)$ form an increasing sequence, so it is sufficient to show that $\mu_{n^2+1}(D_\times^2) \to \infty$.

Fix $n \in \mathbb{N}$, we will compute a lower bound for $U_{D_\times^2}(\{\psi_k \otimes \phi_l\}_{1\le k,l\le n})$, which in turns gives a lower bound for $\mu_{n^2+1}(D_\times^2)$. Since $\{\psi_k \otimes \phi_l\}_{(k,l)\in\mathbb{N}\times\mathbb{N}}$ is a complete set any element of $\mathrm{dom}(D_\times^2)$ is a limit of a sequence of finite linear combinations of the $\psi_k \otimes \phi_l$. This leads us to consider

$$\begin{aligned}\langle \psi_k \otimes \phi_l, D_\times^2(\psi_k \otimes \phi_l)\rangle =& \langle \psi_k \otimes \phi_l, (T^2 \otimes 1_{\mathbb{C}^2})\psi_k \otimes \phi_l\rangle \\ &+ \langle \psi_k \otimes \phi_l, \left(\frac{1}{(1+s)^2} \otimes \left(-\partial_\theta^2\right)\right) \psi_k \otimes \phi_l\rangle \\ &+ \langle \psi_k \otimes \phi_l, \left(-i\gamma^1 \frac{1}{(1+s)^2} \otimes \gamma^1\partial_\theta\right) \psi_k \otimes \phi_l\rangle, \\ =& \lambda_k^2 + \langle \psi_k, \frac{1}{(1+s)^2}\psi_k\rangle \nu_l^2 + \langle \psi_k, -i\gamma^1 \frac{1}{(1+s)^2}\psi_k\rangle\langle \phi_l, \gamma^1\partial_\theta\phi_l\rangle, \\ =& \lambda_k^2 + \left|\left|\frac{1}{1+s}\psi_k\right|\right|^2 \nu_l^2 + \nu_l\langle \psi_k, -i\gamma^1 \frac{1}{(1+s)^2}\psi_k\rangle\langle \phi_l, \gamma^3\phi_l\rangle.\end{aligned}$$

While no longer obvious from this last expression, this is in fact a real number. Moreover, since $\frac{1}{1+s}$ is bounded below by $\frac{1}{1+\varepsilon}$ on $(-\varepsilon, \varepsilon)$ we find that

$$\langle \psi_k \otimes \phi_l, D_\times^2 \psi_k \otimes \phi_l\rangle \geq \lambda_k^2 + \frac{1}{(1+\varepsilon)^2}\nu_l^2 - \frac{1}{(1-\varepsilon)^2}\nu_l,$$

where we used Cauchy-Schwartz to find a lower bound for the rightmost term.

Every element Ψ of $\mathrm{dom}(D_\times^2)$ can be written

$$\Psi = \sum_{k,l=0}^{\infty} \alpha_{(k,l)}\psi_k \otimes \phi_l$$

since the $\{\psi_k \otimes \phi_l\}$ form a complete orthonormal basis. Moreover $||\Psi||^2 = \sum_{k,l=0}^{\infty} |\alpha_{(k,l)}|^2$ and $\Psi \perp \phi_k \otimes \phi_l$ if and only if $\alpha_{(k,l)} = 0$. Then

$$\langle \Psi, D_\times^2 \Psi\rangle = \sum_{k,l=0}^{\infty} |\alpha_{(k,l)}|^2 \langle \psi_k \otimes \phi_l, D_\times^2 \psi_k \otimes \phi_l\rangle.$$

since $D_\times \Psi = \sum \alpha_{(k,l)} D_\times \psi_k \otimes \phi_l$.

If Ψ is an admissible element in the infimum of $U_{D_\times^2}(\{\psi_k \otimes \phi_l\}_{1\leq k,l\leq n})$, then $\alpha_{(k,l)} = 0$ for $k, l \leq n$ and $\sum |\alpha_{(k,l)}|^2 = 1$, which means

$$\langle \Psi, D_\times^2 \Psi\rangle \geq \lambda_{n+1}^2 + \frac{1}{(1+\varepsilon)^2}\nu_{n+1}^2 - \frac{1}{(1-\varepsilon)^2}\nu_{n+1}.$$

The right hand side of this equation tends to infinity as n tends to infinity, so $\mu_{n^2+1}(D_\times^2) \to \infty$ as desired. $\square$

7.4 The Kasparov product of $\widetilde{\iota_!}$ and $[\mathbb{R}^2]$

According to Proposition 7.5 the data $(L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2), D_\times; 1 \otimes \gamma^3 \otimes \gamma^3)$ defines an unbounded spectral triple. We constructed this cycle using a connection from the two cycles $(\tilde{\mathcal{E}}, \tilde{S}; \gamma^3)$ and $(L^2(\mathbb{R}^2), D_{\mathbb{R}^2}; \gamma^3)$ with the intention that it represented their product, inspired by [KL13].

Let us now verify that we have indeed accomplished this. We will prove this using Kucerovsky's criterion (Theorem 4.8).

Proposition 7.18. *The $KK_0(C(S^1), \mathbb{C})$ cycle $(L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, D_\times; 1 \otimes \gamma^3 \otimes \gamma^3)$ represents the product of $\widetilde{\iota_!}$ and $[\mathbb{R}^2]$.*

Proof. Throughout this proof we will use $D = V^* D_\times V$ and we suppress the use of the unitary U in Proposition 7.2.

We need to check the three conditions of Kucerovsky. The connection condition is automatic by Lemma 4.15 and Lemma 5.2.

In the notation of Definition 4.6 set $\mathcal{W} = C_c^\infty(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, this is dense and preserved by both $(\mu_1 i + \tilde{S})^{-1}$ and $(\mu i + D)^{-1}$. Since D is defined on $\mathcal{W}$, this proves that the resolvent of $\tilde{S}$ is compatible with D.

Finally we consider the positivity condition. Using symmetry of $\tilde{S}$ and D we find that we want to prove that

$$\langle \psi, ((\tilde{S} \otimes 1)D + D(\tilde{S} \otimes 1))\psi \rangle \geq C \langle \psi, \psi \rangle,$$

holds on $C_c^\infty(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ for some $C \in \mathbb{R}$. Using the (anti) commutation properties of the γ-matrices, we find that

$$\begin{aligned}
\langle \psi, ((\tilde{S} \otimes 1)D + D(\tilde{S} \otimes 1))\psi \rangle &= \langle \psi, (2f(s)^2 \otimes 1 \otimes 1 + -if'(s) \otimes \gamma^2\gamma^3 \otimes \gamma^1)\psi \rangle, \\
&= \langle \psi, (2f(s)^2 \otimes 1 \otimes 1 + f'(s) \otimes \gamma^1 \otimes \gamma^1)\psi \rangle, \\
&= \langle \psi, f(s)^2 \psi \rangle + \langle \psi, (f(s)^2 \otimes 1 \otimes 1 + f'(s) \otimes \gamma^1 \otimes \gamma^1)\psi \rangle, \\
&\geq \langle \psi, \psi \rangle + \langle \psi, (f(s)^2 - f'(s))\psi \rangle, \\
&= \langle \psi, \psi \rangle - \alpha^2 \langle \psi, \psi \rangle,
\end{aligned}$$

so we may choose $C = 1 - \alpha^2$.

All three conditions of Kucerovsky are satisfied, so $(L^2, D_\times)$ indeed represents the product of $\widetilde{\iota_!}$ and $[\mathbb{R}^2]$. □

7.5 The Kasparov Product of S^1 and the Index Class

We will now verify that the cycle $(L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, D_\times; 1 \otimes \gamma^3 \otimes \gamma^3)$ also represents the product of $\widetilde{[S^1]}$ and $\mathbb{1}$, again by applying Kucerovsky's criterion.

Before we do so, let us recap the domains of (essential) self-adjointness of our operators. Recall that we can write

$$D_\times = \gamma^3 \otimes T + i\gamma^2 \partial_\theta \otimes \frac{1}{1+s}$$

as operator on $L^2(S^1, \mathbb{C}^2) \otimes L^2((-\varepsilon, \varepsilon), \mathbb{C}^2)$. By Corollary 7.12 $D_\times$ is essentially self-adjoint on the domain $\mathrm{dom}(i\gamma^2\partial_\theta) \otimes_{alg} \mathrm{dom}(T)$. Since T is essentially self-adjoint with domain $C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$, according to Proposition 6.2, and $i\gamma^2\partial_\theta$ is essentially self-adjoint on $C^\infty(S^1, \mathbb{C}^2)$, Corollary 7.13 implies that $D_\times$ is also essentially self-adjoint on $C^\infty(S^1, \mathbb{C}^2) \otimes_{alg} C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$.

Proposition 7.19. *The $KK_0(C(S^1), \mathbb{C})$ cycle $(L^2(S^1 \times (-\varepsilon, \varepsilon)) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, D_\times; 1 \otimes \gamma^3 \otimes \gamma^3)$ represents the product of $[S^1]$ and $\mathbb{1}$.*

Proof. We will first check the connection condition. In accordance to the notation of Theorem 4.8 and to avoid confusion between T and T_ξ we will write D_2 for the operator T of the index class. So we have

$$\begin{aligned} D_1 &= i\gamma^2\partial_\theta, \\ D_2 &= i\gamma^1\partial_s + \gamma^2 f(s), \\ D_\times &= \left(D_1 \otimes \frac{1}{1+s}\right) + (\gamma^3 \otimes D_2). \end{aligned}$$

We will verify that the graded commutator

$$\left[\begin{pmatrix} D_\times & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_\xi \\ T_\xi^* & 0 \end{pmatrix}\right]$$

is bounded on $(C^\infty(S^1, \mathbb{C}^2) \otimes_{alg} C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)) \oplus C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$ for $\xi \in C^\infty(S^1, \mathbb{C}^2)$.

Let $\xi \in C^\infty(S^1, \mathbb{C}^2)$ and $\psi \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$, then

$$\begin{aligned} (D_\times T_\xi - (-1)^{\partial\xi} T_\xi D_2)\psi &= D_\times(\xi \otimes \psi) - (-1)^{\partial\xi} \xi \otimes D_2\psi, \\ &= D_1\xi \otimes \frac{1}{1+s}\psi + (-1)^{\partial\xi} \xi \otimes D_2\psi - (-1)^{\partial\xi} \xi \otimes D_2\psi, \\ &= D_1\xi \otimes \frac{1}{1+s}\psi. \end{aligned}$$

The operator norm of $D_\times T_\xi - (-1)^{\partial\xi} T_\xi D_2$ is then bounded by $\frac{1}{1-\varepsilon}||D_1\xi||$.

Now for the other component, let again $\xi, \phi \in C^\infty(S^1, \mathbb{C}^2)$ and $\psi \in C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$.

Then

$$
\begin{aligned}
(T_\xi^* D_\times - (-1)^{\partial\xi} D_2 T_\xi^*)(\phi \otimes \psi) &= T_\xi^*(D_1\phi \otimes \frac{1}{1+s}\psi + \gamma^3\phi \otimes D_2\psi) \\
&\quad - (-1)^{\partial\xi} D_2(\langle \xi, \phi\rangle_{L^2}\psi), \\
&= \langle \xi, D_1\phi\rangle_{L^2}\frac{1}{1+s}\psi + \langle \xi, \gamma^3\phi\rangle_{L^2} D_2\psi \\
&\quad - (-1)^{\partial\xi}\langle \xi, \phi\rangle_{L^2} D_2\psi, \\
&= \langle D_1\xi, \phi\rangle_{L^2}\frac{1}{1+s}\psi.
\end{aligned}
$$

Hence the operator norm of $T_\xi^* D_\times - (-1)^{\partial\xi} D_2 T_\xi^*$ is also bounded by $\frac{1}{1-\varepsilon}||D_1\xi||$.

Putting this together we find that the connection condition is satisfied, since it is satisfied on a core.

Next up is the compatibility condition. We use the same strategy as in Proposition 7.18 and choose $\mathcal{W} = C^\infty(S^1, \mathbb{C}^2) \otimes_{alg} C_c^\infty((-\varepsilon, \varepsilon), \mathbb{C}^2)$. Since both resolvents preserve smoothness and $D_\times$ is defined on the tensor product of the smooth functions this shows compatibility.

Finally we want to show the positivity condition. Again we proceed similarly to Proposition 7.18 and show that

$$
\langle \phi \otimes \psi, ((D_1 \otimes 1)D_\times + D_\times(D_1 \otimes 1))(\phi \otimes \psi)\rangle \geq C\langle \phi \otimes \psi, \phi \otimes \psi\rangle
$$

Since $D_1 \otimes 1$ anti-commutes with $\gamma^3 \otimes D_2$ this term drops out, while $D_1 \otimes 1$ commutes with $D_1 \otimes \frac{1}{1+s}$ to give

$$
\begin{aligned}
\langle \phi \otimes \psi, ((D_1 \otimes 1)D_\times + D_\times(D_1 \otimes 1))(\phi \otimes \psi)\rangle &= 2\langle \phi \otimes \psi, D_1^2\phi \otimes \frac{1}{1+s}\psi\rangle, \\
&= 2\langle D_1\phi, D_1\phi\rangle\langle \psi, \frac{1}{1+s}\psi\rangle, \\
&\geq \frac{2}{1-\varepsilon}||D_1\phi||^2||\psi||^2 \geq 0.
\end{aligned}
$$

Thus we have verified all the conditions in Theorem 4.8, so our product cycle represents the product of $[S^1]$ and $\mathbb{1}$. □

Corollary 7.20. *At the level of KK-classes we have $\widetilde{[S^1]} = \widetilde{\iota_!} \otimes_{C_0(\mathbb{R}^2)} [\mathbb{R}^2]$.*

8 Discussion

In this thesis we investigated the shriek class corresponding to the embedding $\iota : S^1 \hookrightarrow \mathbb{R}^2$. While this embedding is probably the simplest non-trivial example, it led to interesting results. We have proven that $\widetilde{[S^1]} \otimes \mathbb{1} = \widetilde{\iota_!} \otimes [\mathbb{R}^2]$ using Kucerovsky's criterion and constructed an explicit unbounded representative for this product. Essential for this was the construction of an index class that represents $\mathbb{1} \in KK_0(\mathbb{C}, \mathbb{C})$. This allowed us to reduce the "large" object $\tilde{\iota}_! \otimes [\mathbb{R}^2]$ to the "small" object $\widetilde{[S^1]}$.

During our analysis of the various KK-classes, we made several choices. Most obviously the choice $S = \alpha \tan(\alpha s)$. This choice for $\tan(x)$ is motivated in Section 6.1 with the argument that it allows us to solve the differential equation for the integrating factor. However, we would like to state the following conjecture.

Conjecture 1. *For any $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ that tends to $\pm\infty$ at $\pm\varepsilon$ "quickly enough", $(L^2((-\varepsilon, \varepsilon), \mathbb{C}^2), T)$ is an unbounded representative for $\mathbb{1} \in KK_0(\mathbb{C}, \mathbb{C})$.*

A more subtle, and perhaps more interesting, choice is our definition of $\mathcal{E}$ as functions on $S^1 \times (-\varepsilon, \varepsilon)$ rather than functions on $S^1 \times \mathbb{R}$. It is the finite length of the interval $(-\varepsilon, \varepsilon)$ that disqualifies the use of the work in [KL13], as discussed in Remark 7.6.

It might be interesting to investigate in detail what happens if we take our fibres as $\mathbb{R}$ instead of $(-\varepsilon, \varepsilon)$. This would allow us to use the function x rather then $\tan(x)$ for S, which would make T_0 very similar to (a combination of) ladder operators for the quantum mechanical oscillator. However, since we need the fibre to map to a $(-\varepsilon, \varepsilon)$-neighbourhood of $S^1 \subset \mathbb{R}^2$ we would need to introduce measures into the various Hilbert spaces and unitaries in Section 7.1.

The presence of these measures might stop us from applying the results of [KL13], since they would effectively "squash" x into a function that tends to infinity after a finite distance. This makes us not hopeful that this approach will yield simpler results.

There is a noticeable difference with the result obtained in [KS16] for submersions, since the "factorization" of Dirac operators that we find contains a rather non-trivial term corresponding to the Index class, or unit in $KK_0(\mathbb{C}, \mathbb{C})$. Especially when generalizing to immersions with codimension greater than 1, careful analysis is needed to properly identify these terms.

The eventual goal of this research project is to find an unbounded representative of the shriek class of an arbitrary immersion. A first step towards this would be to find an unbounded representative for the shriek class of any codimension

1 embedding. Specifically an immediate next step could be to investigate the embedding $S^2 \hookrightarrow \mathbb{R}^3$, since this is a nice computable setting with a very similar flavour as the example we computed in this thesis.

The example $S^2 \hookrightarrow \mathbb{R}^3$ would also be interesting since we expect an obstruction corresponding to the curvature of the immersed manifold to appear in the factorization of the Dirac operator, similar to [KS16].

After this example we might work on general codimension 1 embeddings, using for example the formulas in [Bär96] which relate Dirac operators on hypersurfaces to the Dirac operator on their ambient manifold. The main ingredients that need to be generalized for this application is the definition of the inner product on $\mathcal{E} = C_0(S^1 \times (-\varepsilon, \varepsilon))$, where the factor $\frac{1}{r}$ probably should be replaced by the mean curvature of the embedding, and the connection on $\mathcal{E}$, which should be modified to remain metric for the new inner product.

Initially it might be useful to restrict to compactly embedded manifolds, since then we can always find an $\varepsilon > 0$ such that an ε-neighbourhood of the 0-section is diffeomorphic to a neighbourhood of the embedded manifold where the mean curvature of the embedding remains bounded. This allows S to be independent of the point on the embedded manifold, we used this, for example, in Corollary 7.12. Boundedness of the curvature term plays an important role in various parts of this thesis, usually in the guise of $\frac{1}{1+s}$ being bounded above and below on $(-\varepsilon, \varepsilon)$.

When this is accomplished we can try to consider general embeddings, central issues here are that we need a K-orientation as in [CS84] which makes our shriek module $(\mathcal{E}, S)$ higher dimensional. While $\mathcal{E}$ can likely still be modelled after the normal bundle, the action of S on this normal bundle merits some thought, although we can use [CS84] for inspiration, similar to what we did in Section 5.2.

Finally we would like to treat arbitrary immersions. While this likely requires some more work, it should not be hard since most constructions we use are local rather than global.

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